Dividing by a fraction: Why do we invert and multiply?

Consider a division problem like $\frac{a}{b} \div \frac{c}{d}$. We want to show that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

Let's start with the original problem:

$$\frac{a}{b} \div \frac{c}{d}$$

This can be rewritten as a fractional division (a complex fraction) like this:

$$= \frac{\frac{a}{b}}{\frac{c}{d}}$$

This fraction can be multiplied by 1 without changing the value:

$$= \frac{\frac{a}{b}}{\frac{c}{d}} \cdot 1$$

As you know, 1 can look like a lot of things. We need to be creative as to our choice of 1. In this case, let's creatively choose 1 to be $\frac{\frac{d}{c}}{\frac{d}{c}}$ since $\frac{d}{c}$ is the reciprocal of $\frac{c}{d}$, the denominator:

$$= \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{\frac{d}{c}}{\frac{d}{c}}$$

We can multiply these two fractions together:

$$= \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}}$$

As intended the denominator is a product of reciprocals (multiplicative inverses) so their product is 1:

$$= \frac{\frac{a}{b} \cdot \frac{d}{c}}{1}$$

Last, any expression divided by 1 is itself (the numerator):

$$= \frac{a}{b} \cdot \frac{d}{c}$$

The original expression, $\frac{a}{b} \div \frac{c}{d}$ is equal to this result. This means that $\frac{a}{b} \div \frac{\mathbf{c}}{d} = \frac{a}{b} \cdot \frac{d}{\mathbf{c}}$.

In other words, when dividing by a fraction, we must "invert and multiply."