DIVISIBILITY RULE FOR 7 (AND 13, AND ...)

by Robert H. Prior, March 4, 2005

Divisibility Rule for 7: A whole number, N, is a multiple of 7 if the following procedure leads to another multiple of 7*:

(1) Subtract the ones digit from N,

(2) Dividing the result by 10, and

(3) Subtract—from *that* result—twice the *original* ones digit.

*the multiple of 7 may be 0 or negative.

This is best explained in an example. Consider 7 x 39 = 273, so 273 is a multiple of 7. Applying the Divisibility Rule for 7 to 273 yields the following:

(1) Subtract the ones digit from N: Subtract 3 from 273:

$$\begin{array}{r} 273 \\ -\ \ 3 \\ \hline 270 \end{array}$$

(2) Dividing the result by 10: Divide 270 by 10: $270 \div 10 = 27$

(3) Subtract—from *that* result—twice the *original* ones digit: Subtract, from 27, twice 3:

$$\begin{array}{r} 27 \\ -\ 6 \\ \hline 21 \end{array}$$

a multiple of 7 → 21

Since this procedure leads to a multiple of 7, it must be that the original number, 273, is a multiple of 7, as shown in this long division:

$$\begin{array}{r} 39 \\ 7\overline{)273} \\ -\ 21 \\ \hline 63 \\ -\ 63 \\ \hline 0 \end{array}$$

A much quicker way to use the procedure is shown here

$$\begin{array}{r} 2\ 7\ \not{3} \\ -\ \ \ 6 \\ \hline 2\ 1 \end{array} \quad \times 2$$

2 1 ← Since 21 is a multiple of 7, it must be that 273 is a multiple of 7.

Let's apply this more direct procedure to a larger multiple of 7:   7 x 568 = 3,976:

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        x 2
  3 9 7 6
–   1 2
  -------
  3 8 5    ←  We don't know if 385 is a multiple of 7
              or not, so we continue the procedure.
      x 2
  3 8 5
– 1 0
  -----
  2 8      ←  Since 28 is a multiple of 7, it must be that
              both 385 and 3,976 are multiples of 7.
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What is the proof behind this procedure?

Consider, without loss of generality, a three digit number, $N = 100a + 10b + c$, where a, b, and c are single digits, $a \neq 0$.

Applying the Divisibility Rule for 7 to $100a + 10b + c$, we get:

(1) Subtract c: $100a + 10b$

(2) Divide by 10: $10a + b$

(3) Subtract twice c: $10a + b - 2c$

Claim: $100a + 10b + c$ is a multiple of 7 if and only if $10a + b - 2c$ is a multiple of 7

P ⇆ **Q**

Proof: (1) Prove P if Q: (a, b, and c are single digits and d is an integer)

Assume: $10a + b - 2c$ is a multiple of 7;

Show: $100a + 10b + c$ is a multiple of 7: $10a + b - 2c = 7d$

Multiply each side by 10: $100a + 10b - 20c = 70d$

Add 20c to each side: $100a + 10b = 70d + 20c$

Add 1c to each side: $100a + 10b + c = 70d + 21c$

The right side has a factor of 7: $100a + 10b + c = 7(10d + 3c)$

∴ $100a + 10b + c$ is a multiple of 7. QED$_1$

We must also show that the converse is true:

Proof: (2) Prove if P then Q:

Assume: $100a + 10b + c$ is a multiple of 7;

Show: $10a + b - 2c$ is a multiple of 7: $100a + 10b + c = 7d$

Subtract c from each side: $100a + 10b = 7d - c$

Divide each side by 10: $10a + b = \frac{7d - c}{10}$

Subtract 2c from each side: $10a + b - 2c = \frac{7d - c}{10} - 2c$

Simplify the right side: $10a + b - 2c = \frac{7d - c}{10} - \frac{20c}{10}$

$10a + b - 2c = \frac{7d - 21c}{10}$

The right side has a factor of 7: $10a + b - 2c = \frac{7(d - 3c)}{10}$

Since a, b, and c are single digits, $10a + b - 2c$ is an integer, not a decimal, so the division by 10 is inconsequential to showing that this number is a multiple of 7

$\therefore$ $10a + 1b - 2c$ is a multiple of 7. QED$_2$

This proof also indicates that, if the process leads to a number that is not a multiple of 7, then the original number is also not a multiple of 7.

Exercise: Determine which of the following numbers are multiples of 7.

1. 959 **2.** 6,152 **3.** 12,845 **4.** 186,137

The divisibility rules for 13, 17, and 37 (and others) are similar to the Divisibility Rule for 7. For each, we subtract the ones digit and then divide by 10. From that result, we subtract a multiple of the original ones digit.

In the Divisibility Rule for 7, the ***subtraction multiplier*** is 2. In other words, in the last step of the process, we subtract **2 times** the original ones digit.

For 13, the divisibility rule has a subtraction multiplier of **9**:

Divisibility Rule for 13: A whole number, N, is a multiple of 13 if the following procedure leads to another multiple of 13:

(1) Subtract the ones digit from N,

(2) Dividing the result by 10, and

(3) Subtract—from that result—**9 times** the *original* ones digit.

Let's test 273: 13 x 21 = 273,
so 273 is a multiple of 13.

x 9

$$\begin{array}{r} 27\not{3} \\ -\,27 \\ \hline 0 \end{array}$$

Since 0 is a multiple of 13, it must be that 273 is a multiple of 13.

Exercise: Determine which of the following numbers are multiples of 13.

5. 728 **6.** 1,911 **7.** 14,238 **8.** 305,734

Interestingly enough, the Divisibility Rule for 7 can use *9* as a subtraction multiplier, too.

Exercise: Determine which of the following numbers are multiples of 7. This time, use 9 as the subtraction multiplier

9. 959 **10.** 6,152 **11.** 12,845 **12.** 186,137

What is the key to recognizing the subtraction multiplier? Consider the end of the second proof of the Divisibility Rule for 7. On the right side, before factoring out 7, we got

$$\frac{7d - 21c}{10}$$

We are able to factor out 7 because the **21c** appeared after getting common denominators:

$$\frac{7d - 1c}{10} - 2c$$

$$= \frac{7d - 1c}{10} - \frac{20c}{10}$$

$$= \frac{7d - 21c}{10} \leftarrow$$

The term 1c was already in the right side numerator, due to the initial subtraction of the ones digit and the subsequent division by 10. The *c* term became a multiple of 7 when we added - 20c to it.

If we did a proof for the Divisibility Rule for 13, we would eventually get to the step

$$\frac{13d - 1c}{10} - \mathbf{???}c$$

The question becomes, "What is the mystery coefficient of the c term that is being subtracted?" Whatever that mystery coefficient is, it is the *subtraction multiplier* for the Divisibility Rule for 13.

In getting common denominators, we'll need to multiply the mystery coefficient by 10; therefore, it needs to be the tens digit of a multiple of 13 that ends in 1 , namely **9**1. (91 = 7 x 13)

Since 91 is a multiple of both 7 and 13, **9** is a subtraction multiplier for both 7 and 13. Of course, 7 is also a factor of **2**1, so it also has a subtraction multiplier of 2, as we originally saw.

This divisibility test procedure works similarly for both 17 and 37. Each has its own subtraction multiplier. Can you discover what they are? Though both 9 and 11 have other tests for divisibility, this procedure also works for them as well, each with its own subtraction multiplier.

Exercise: Determine the subtraction multiplier, used in this divisibility test procedure, for each number.

13. 17 **14.** 37 **15.** 9 **16.** 11

An arithmetic textbook, published in 1874, called *Complete Arithmetic*, by Daniel W. Fish, says this about the divisibility rules for 7, 11 and 13:

"Any number is divisible ...

9. By 7, 11, and 13 if it consists of but *four* places, the first and fourth being occupied by the same significant figures, and the second and third by ciphers."

(A cipher is a 0.) It continues:

"Thus, 2002, 3003, and 5005 are divisible by 7, 11, and 13."

Exercise: Use the techniques demonstrated in this paper to show that each of these is a multiple of 7, 11, and 13.

17. 2002 **18.** 3003 **19.** 5005 **20.** 7007

Will this divisibility rule for 7, 11, and 13 stay true if the first and fourth "places" are two digit numbers?

Let's look: Consider a "four" digit number $N = 1000a + 0b + 0c + a$.

We can apply the rule for both 7 and 13 at the same time using a subtraction multiplier of 9:

(i) subtract the ones digit: $1000a$

(ii) divide the result by 10: $100a$

(iii) subtract 9 times the ones digit: $100a - 9a = 91a$

The result, 91a, is a multiple of both 7 and 13, so N is a multiple of both 7 and 13.

We can apply the rule for 11 using a subtraction multiplier of 1:

(i) subtract the ones digit: $1000a$

(ii) divide the result by 10: $100a$

(iii) subtract 1 times the ones digit: $100a - 1a = 99a$

The result, 99a, is a multiple of 11, so N is a multiple of 11 as well.

Exercise: **21.** Since 99a is a multiple of 9, is it necessary that N a multiple of 9 as well? Explain.

Exercise: **22.** If we were to use a = 13, the "four" digit number (13)00(13) would have to be written as 13,013. Is this number a multiple of 7, 11 and 13?

I hope you enjoyed this tour. You can find more math stuff at http://bobprior.com/forteachers.html