THE ORDER OF OPERATIONS: MATH OR MYTH

By Robert H. Prior, revised March 3, 2005

THE ORDER OF OPERATIONS

The order of operation is a hierarchy that indicates which operation in an expression should be applied first. After the first operation has been applied, the order of operations then addresses which operation to apply next, and so on. Here is the hierarchy, represented by **rank**; the *highest* rank is the 1st rank.

1st rank: Apply any operation within grouping symbols first, keeping this hierarchy in mind should there be more than one operation or other grouping symbols.

2nd rank: Apply any exponents to their respective bases.

3rd rank: Apply multiplication and division, working from left to right. Multiplication and division have equal rank, and if both appear in the expression, then preference is given to the leftmost one.

4th rank: Apply addition and subtraction, working from left to right. In general addition and subtraction have equal rank, and if both appear in the expression, then preference is given to the leftmost one. However, when we get to this level in the hierarchy, the commutative and associative properties allow us to combine terms in any order that works most efficiently.

DOES IT NEED TO BE THIS WAY?

Who or what established the order of operations? Does it really need to be this way, or did some ancient council of mathematicians confer and decide, "This is the way it's going to be"? Is it just a matter of convention that—for the sake of being consistent—the order of operations is applied that way it is?

First, Grouping symbols create a quantity, a single value, and that single value must be known before we can apply any "outside" operations to it. Hence, grouping symbols must have the highest rank.

Second, The exponent is an abbreviation for repeated multiplication. A whole number exponent indicates the number of factors of the base.

For example, $5^3 = 5 \cdot 5 \cdot 5$, *three* factors of 5.

This means that every exponent expression can be expanded to multiplication, thereby eliminating the 2nd rank.

Third, Multiplication is an abbreviation for repeated addition.

For example, $3 \cdot 5$ is the sum of three fives: $5 + 5 + 5$.

Also, division is another form of multiplication, and every quotient can be expressed as a product. For example, $4 \div 2 = 4 \cdot \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$.

This means that every product—and every quotient—can be expanded to addition, thereby eliminating the 3rd rank.

Last, Subtraction is another form of addition. Every difference can be written as a sum.

For example, $7 - 2 = 7 + (-2)$.

So, though it may look rather complicated, most numerical expressions can eventually be expressed in terms of only addition before being evaluated.

So, does the order of operations need to have the hierarchy that it does? In a word, yes!

Example: Write each expression using only addition before being evaluated.

(a) $6 \div 3$

$= 6 \cdot \frac{1}{3}$

$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$

$= 1 + 1$

$= 2$

(b) $3^2 - 5$

$= 3 \cdot 3 - 5$

$= (3 + 3 + 3) + (-5)$

$= 9 + (-5)$

$= 4$

(c) $4 \cdot \frac{3}{2}$

$= 4 \cdot (3 \div 2)$

$= 4 \cdot \left(3 \cdot \frac{1}{2}\right)$

$= (4 \cdot 3) \cdot \frac{1}{2}$ Associative property

$= (3 + 3 + 3 + 3) \cdot \frac{1}{2}$

$= 12 \cdot \frac{1}{2}$

$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$= 1 + 1 + 1 + 1 + 1 + 1$

$= 6$

(d) $10 - 4 \cdot 2$

$= 10 - (2 + 2 + 2 + 2)$

$= 10 - 8$

$= 10 + (-8)$

$= 2$

e) $\dfrac{4 \cdot 2 - 3}{3^2 + 1}$

$= \dfrac{2 + 2 + 2 + 2 - 3}{3 \cdot 3 + 1}$

$= \dfrac{2 + 2 + 2 + 2 + (-3)}{3 + 3 + 3 + 1}$

$= \dfrac{5}{10}$ There's nothing more to do with this except write it as a decimal, 0.5, or as a simplified fraction, $\dfrac{1}{2}$.

$\dfrac{5}{10}$ could be written as $5 \cdot \dfrac{1}{10}$ and then written as $\dfrac{1}{10} + \dfrac{1}{10} + \dfrac{1}{10} + \dfrac{1}{10} + \dfrac{1}{10}$, but we can't go much further than this.

THE MAIN OPERATION

The **main operation** is the operation in the expression that is to be applied last, according to the order of operations. The main operation is used in solving linear equations and in translating numerical and algebraic expressions into, and from, English.

Examples: Identify the main operation in each expression.

a) $-6 + 12 \div 3$

Addition (sum)

b) $5^2 - 8$

Subtraction (difference)

c) $24 \div 6 \cdot 2$

Multiplication (product)

d) $|4 - 15 \div 3| \cdot 8$

Multiplication (product)

e) $\dfrac{5^2 + 7}{3^2 - 1}$

Division (quotient)

f) $|4 - 15 \div 3|$

Absolute value

g) $(9 - 5)^2$

Square

h) $\sqrt{24 + 6 \cdot 2}$

Square root

h) $10^2 - [3 - 2(5 + 4)]$

Subtraction (difference)

In translating from English to arithmetic, if an expression has two or more operations, it is the main operation that is written first in English.

For example, in translating, "The <u>sum</u> of the square of 5 <u>and</u> the product of 2 and 7,"

the first operation mentioned is addition, in the word "sum." This means we'll place a plus sign between to smaller expressions. The smaller expressions, the additive operands, will be "the square of 5" on the left and "the product of 2 and 7" on the right.

The expression looks like this: $5^2 + 2 \cdot 7$

Exercise: Translate from English to an algebraic expression. Let x represent "a number."

1. The sum of a number and its square.

2. Twice the sum of a number and 9.

3. The product of a number and the difference between the number and 4.

4. The square root of the sum of 3 and the quotient of a number and 4.

Exercise: Translate each numerical expression into English.

5. $2x - 5$

6. $(4x + 3)^2$

7. $x^3(x - 1)$

At the end of the movie *Wizard of Oz,* the scarecrow "proves" his intelligence by quoting the Pythagorean Theorem ... or does he? After receiving his diploma, he recites, "The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side."

Exercise: **6.** Correctly translate the Pythagorean Theorem into English.

The **main operation** is helpful in solving non-quadratic equations. When we eliminate an operation from the one side of an equation—for the purpose of isolating the variable—it is the main operation that is eliminated in each step, and the main operation will change from step to step.

For example, in the equation .... $9 - 7x = -5$

we can isolate the variable by removing both operations multiplication and addition. Which should we eliminate first? Since the main operation is addition, we eliminate it first by adding - 9 to each side (the additive inverse).

$$\begin{aligned} -9 \quad &= -9 \\ \hline -7x &= -14 \\ &\vdots \end{aligned}$$

For example, in the equation .... $\sqrt{3x^3 + 1} = 5$

the main operations is *square root*, and we can eliminate it by squaring each side .... $\left(\sqrt{3x^3 + 1}\right)^2 = (5)^2$

Now the main operation is *addition*: add - 1 to each side .... $3x^3 + 1 = 25$

Now the main operation is *multiplication*: divide each side by 3 .... $3x^3 = 24$

Now the main operation is *cube*: take the cube root of each side .... $x^3 = 8$

and the variable is now isolated .... $x = 2$

Exercise: **7.** See if you can find other uses for the Main Operation.