Graphing Quadratic Functions

A. THE STANDARD PARABOLA

The graph of a quadratic function is called a **parabola**. The most basic graph is of the function $y = x^2$, as shown in *Figure 1*, and it is to this graph which all other parabolas will be compared. I'll refer to it as the ***standard parabola*** or the ***standard graph***.

The standard parabola is, at first, built by plotting points, choosing a few values of *x* (the independent variable) and finding the corresponding value of *y* accordingly. Usually, finding seven points is sufficient to give us a good sense of the shape of the graph: the lowest (or highest) point—called the ***vertex***, three points to the right of the vertex and three points to the left, though not necessarily in that order.

$y = x^2$

x	*y*
- 3	9
- 2	4
- 1	1
0	0
1	1
2	4
3	9

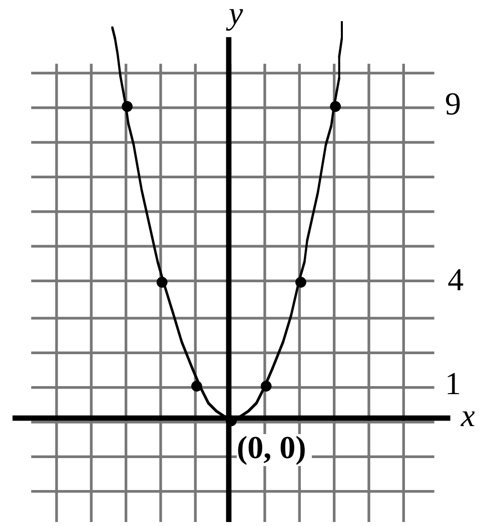
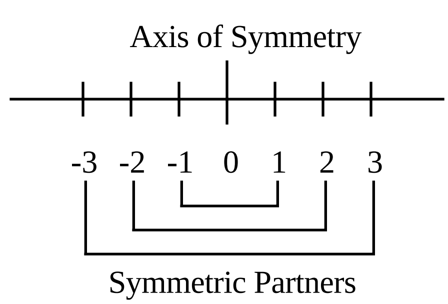


Figure 1

The standard graph of $y = x^2$ has several features common to all parabolas, but these features are more recognizable because of the location of the graph, "centered," as it is, at the origin. Some features of this graph are

1. its vertex is at the origin;
2. it is symmetric about a line, called the *axis of symmetry*; in the standard parabola, the *y*-axis is the axis of symmetry;
3. there are easily recognizable ***symmetric pairs***.



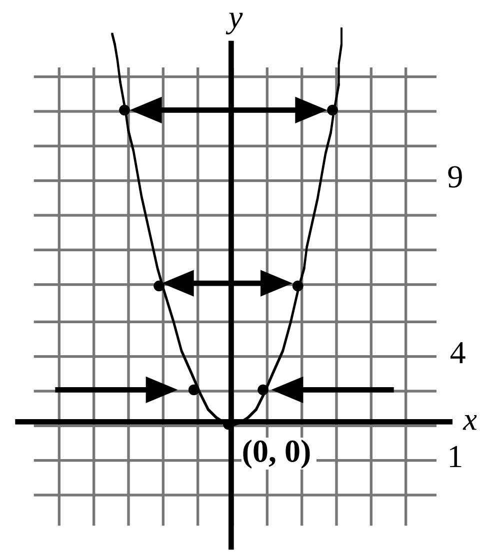


Figure 2

Symmetric Pairs: In the case of the standard parabola, the symmetric pairs shown are -3 and 3; -2 and 2; and -1 and 1. Other symmetric pairs — not shown — include, for example, $-\frac{1}{2}$ and $\frac{1}{2}$, but we will restrict most of the discussion to integers, wherever possible. We can also say that - 3 is the ***symmetric partner*** of + 3. Finally, the line segment drawn between any two symmetric pairs is perpendicular to the ***axis of symmetry***, and the axis of symmetry bisects that line segment.

In $y = x^2$, 0 is the only integer which has no symmetric partner, and it is this value that is the x-coordinate of the vertex. Furthermore, this x-value—being the only point without a symmetric partner—indicates the location of the axis of symmetry, $x = 0$.

Once we identify the vertex (and axis of symmetry), we can then identify three points on the left (or right) side of the axis of symmetry and find the corresponding symmetric partners to the known points.

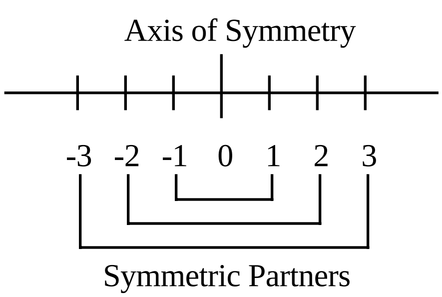
B. The Vertical Shift

Of course, there are many other parabolas, each "starting" at its own vertex. We can generate other parabolas simply by moving the vertex to a new location. Doing so may result in a new axis of symmetry and a new set of symmetric pairs.

Here is the graph of $y = x^2 + 1$. Notice that the symmetric pairs occur at the same x-values as the standard parabola, but the y-values have each increased by 1.

$y = x^2 + 1$

x	y
-3	10
-2	5
-1	2
0	1
1	2
2	5
3	10



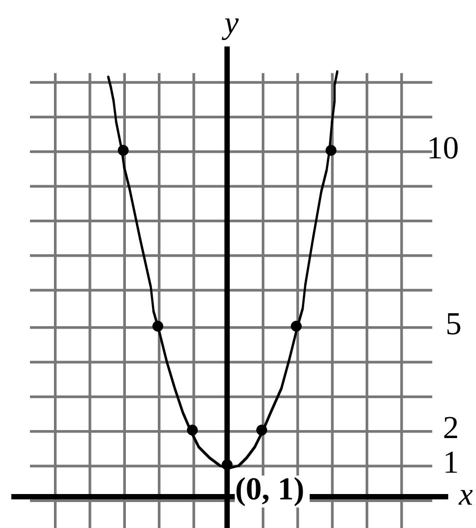
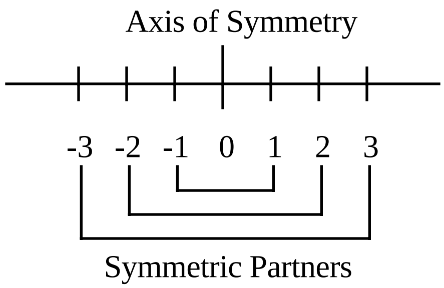


Figure 3

The same is true for the graph of $y = x^2 - 3$ except that the y-values have all decreased by 3 (compared to the standard parabola).

$y = x^2 - 3$

x	y
-3	6
-2	1
-1	-2
0	-3
1	-2
2	1
3	6



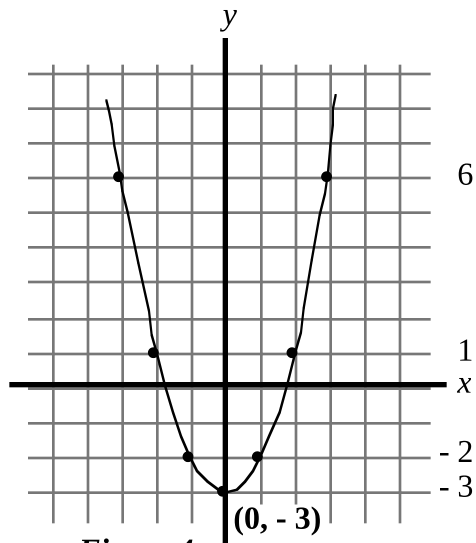


Figure 4

We say that the graphs of $y = x^2 - 3$ and $y = x^2 + 1$ result in a ***vertical shift*** of 3 downward or 1 upward, respectively, from the standard graph. Notice that the constant is *outside of* the squared term.

C. The Horizontal Shift

How can we affect a change—or shift—in the symmetry of the *x*-values? In other words, what needs to be different in the function in order to affect a ***horizontal shift*** of the graph, either to the right or to the left? The change needs to come from within the square, from the *argument* of the square.

For example, in $y = (x - 3)^2$ we must apply “minus 3” to the *x*-value *before* we square to find the corresponding y-value. This means that we’re going to have a different *x*-value without a symmetric partner. To find this new *x*-value—which leads directly to the location of the vertex and the axis of symmetry—we need to simply find where that ***argument is equal to 0***:

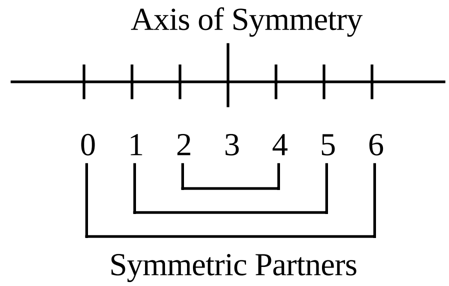
$$x - 3 = 0 \text{ when } x = 3.$$

$x = 3$ becomes the *x*-value that has no symmetric partner. This means that the vertex will be located at $x = 3$, and the axis of symmetry is the vertical line $x = 3$. Having the vertex at a new location does, indeed, yield a new set of symmetric partners. Instead of being centered around 0, they are centered around 3.

Also, because of the symmetry, we need not do all of the work to find seven points on the parabola (as we did previously). We can find the vertex point and then three points to its immediate right. By symmetry, we can then find the coordinates of their symmetric partners.

x	$y = (x-3)^2$	(x, y)
6	$(6-3)^2 = (3)^2 = 9$	(6, 9)
5	$(5-3)^2 = (2)^2 = 4$	(5, 4)
4	$(4-3)^2 = (1)^2 = 1$	(4, 1)
3	$(3-3)^2 = (0)^2 = 0$	(3, 0)
2	1	(2, 1)
1	4	(1, 4)
0	9	(0, 9)

axis of symmetry



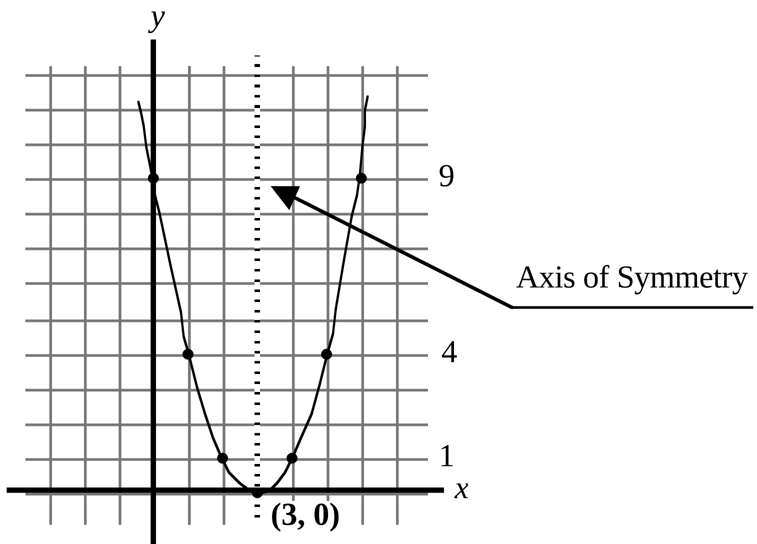


Figure 5

Figure 5

D. SHIFTING THE GRAPH IN TWO DIRECTIONS

So far, each of the parabolas shown graphed has had its vertex on either the *y*-axis or the *x*-axis. It is, of course, possible to have a parabola that has a vertex in one of the quadrants. This means that the function will have one constant added (or subtracted) within the argument — for a horizontal shift — and another constant added (or subtracted) outside of the squared term — for a vertical shift.

A quadratic function that has a graph with both a vertical shift and a horizontal shift can look like this:

$$y = (x - \mathrm{h})^2 + \mathrm{k}$$

In trying to determine the location of the vertex, we rely on the order of operations to indicate where we should begin.

(1) first, within the parentheses, we find the x-value of the vertex by setting the argument of the square equal to 0:

$$x - h = 0$$

$$x = h;$$

(2) second, we find its y-coordinate by replacing x with h:

$$y = (h - h)^2 + k$$

$$y = 0^2 + k$$

$$y = k$$

So, the vertex is the point (h, k).

E. TWO DIFFERENT FORMS OF THE QUADRATIC FUNCTION

To this point, we have seen two forms of a quadratic function:

(i) **Standard form:** $y = x^2 + bx + c$, which is also written as $y = \mathbf{a}x^2 + bx + c$, $a \neq 0$*;

(ii) **Vertex form:** $y = (x - h)^2 + k$; the vertex is (h, k) and the axis of symmetry is $x = h$; vertex form can also be written as $y = \mathbf{a}(x - h)^2 + k$, $a \neq 0$*.

* Throughout the rest of this paper, it will be assumed that $a \neq 0$.

F. FINDING THE ZEROS, THE AXIS OF SYMMETRY AND THE VERTEX

The **zeros** of the parabola are the values of x that make y equal to 0. They are, in other words, the x-intercepts of the graph of the parabola.

For example, we can find the zeros of $y = x^2 - 8x + 12$ by

setting the quadratic expression equal to 0: $x^2 - 8x + 12 = 0$

Factor: $(x - 6)(x - 2) = 0$

$$x - 6 = 0 \quad \text{or} \quad x - 2 = 0$$

$$x = 6 \quad \text{or} \quad x = 2$$

giving us two points on the parabola, (6 , 0) and (2, 0).

Once we have the zeros, we have a symmetric pair of points, and the axis of symmetry must be equidistant from each. The axis of symmetry is found as the average (mean) of the two zeros found:

Axis of Symmetry: $x = \dfrac{6 + 2}{2} = 4$

And once we have the axis of symmetry we can find the vertex:

$$y_{(4)} = (4)^2 - 8(4) + 12$$
$$= 16 - 32 + 12$$
$$= -4$$

The vertex is (4, - 4).

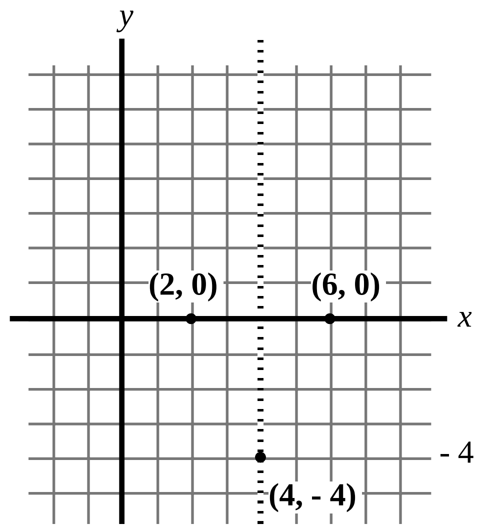


Figure 6

Finding the zeros of a quadratic function that is not factorable, however, requires that we employ one of a few different techniques, possibly *completing the square* or *applying the quadratic formula*.

For example, the trinomial in

$y = x^2 - 4x - 3$

is not factorable (with integer constants). It can be shown to be equivalent to

$y = (x - 2)^2 - 7,$

and we can see from the graph that the zeros are not integers:

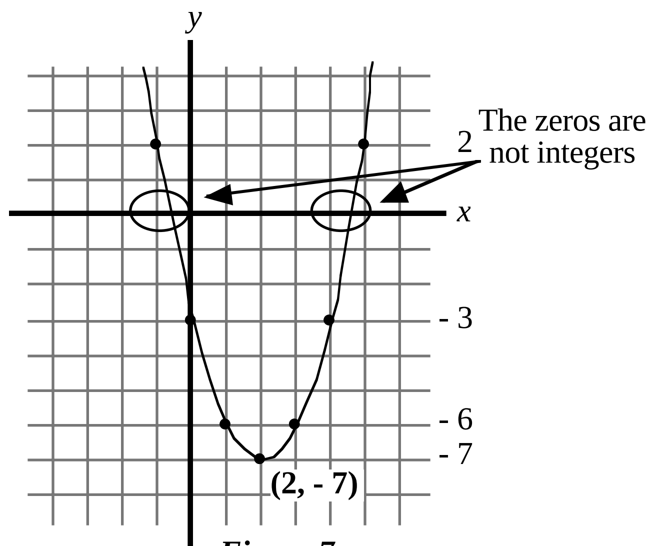


Figure 7

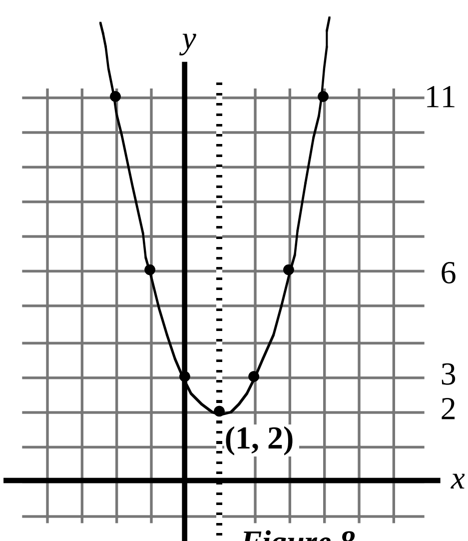


Figure 8

Likewise, the trinomial in $y = x^2 - 2x + 3$ is not factorable, and it turns out, has no zeros (x-intercepts) at all. This quadratic function is equivalent to $y = (x - 1)^2 + 2$, and as shown in the graph, it doesn't intercept the x-axis anywhere.

G. FINDING OTHER POINTS OF THE PARABOLA

When finding the zeros of the function we look at where the graph crosses the x-axis. This could also be stated as where the graph crosses the horizontal line $y = 0$.

We may, at other times, be interested in finding, for example, where the parabola crosses the horizontal line $y = 3$. That would be the same as asking, "What value of x will make the function equal to 3?"

For example, in the function $y = x^2 + 4x + 6$, at what values of x does the graph cross the line $y = 3$?

That means that we are to replace y with 3 and get $3 = x^2 + 4x + 6$

and we can set it to 0 and factor: $0 = x^2 + 4x + 3$

$0 = (x + 3)(x + 1)$

$x = -3$ or $x = -1$

Instead of writing these as solutions to an equation we can say that

"The parabola will cross the line $y = 3$ at $x = -3$ and at $x = -1$."

This also means that we know two points on the parabola, namely (-3, 3) and (-1, 3), and *these two points are a symmetric pair*.

The axis of symmetry can again be found,

Axis of Symmetry: $x = \dfrac{-3 + (-1)}{2} = -2$

leading us to the vertex:

$$y_{(-2)} = (-2)^2 + 4(-2) + 6$$
$$= 4 + (-8) + 6$$
$$= 2;$$

The vertex is (- 2, 2).

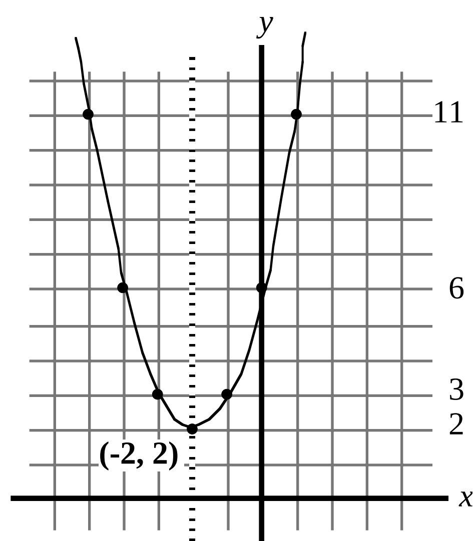


Figure 9

Choosing a constant at random, though, will not always lead to a factorable polynomial. If, however, in $y = ax^2 + bx + c$ we choose to find the **c's** (where the graph crosses the line $y = c$, the trinomial's constant term), we'll be guaranteed to find a factorable binomial..

For example, in the function $y = x^2 + 4x + 6$, let's find, instead, where the graph crosses the horizontal line $y = 6$.

$$x^2 + 4x + 6 = 6$$

$$x^2 + 4x = 0$$

$$x(x + 4) = 0$$

$$x = 0 \quad \text{or} \quad x = -4$$

So, the parabola will cross the line $y = 6$ at $x = 0$ and at $x = -4$. This means that we know two points on the parabola: (0, 6) and (-4, 6), a symmetric pair. Additionally, one of these points, (0, 6), is the y-intercept.

Of course, this symmetric pair leads to the same axis of symmetry as the symmetric pair of (-3, 3) and (-1, 3).

Axis of Symmetry: $x = \dfrac{0 + (-4)}{2} = -2$ and the vertex is (- 2, 2). (see ***Figure 9***)

Example: Consider the quadratic function $y = x^2 + 2x + 5$

(a) find where graph crosses the horizontal line $y = 5$; (b) use those values to find the axis of symmetry; and (c) use the axis of symmetry to find the vertex

(a)

$$x^2 + 2x + 5 = 5$$

$$x^2 + 2x = 0$$

$$x(x + 2) = 0$$

$$\underline{x = 0 \text{ or } x = -2}$$

(b) The axis of symmetry is a horizontal line that is the midpoint between the two zeros:

$$A = \frac{-2 + 0}{2} = \frac{-2}{2} = -1$$

So, the **axis of symmetry** is $x = -1$.

c) To find the vertex we can use - 1 as a replacement value for x in the original function:

$$y_{(-1)} = (-1)^2 + 2(-1) + 5 = 1 - 2 + 5 = 4$$

So, the **vertex** is at the point (- 1, 4).

Finding the c's is more reliable than finding the zeros because not every quadratic function has real zeros, not every parabola has *x*-intercepts. However, every parabola does have a *y*-intercept and, except when the *y*-axis is the axis of symmetry, the *y*-intercept has a symmetric partner. Those two points can always be found by setting the function equal to c.

For the quadratic function $y = ax^2 + bx + c$, where $b \neq 0$, we can develop a general formula for the axis of symmetry by finding where it crosses the horizontal line $y = c$. (If b = 0, then the *y*-axis is the axis of symmetry.)

a)

$$ax^2 + bx + c = c$$

$$ax^2 + bx = 0$$

$$x(ax + b) = 0$$

$$x = 0 \quad \text{or} \quad x = \frac{-b}{a}$$ These are symmetric partners.

b) The axis of symmetry is a horizontal line that is the average of any two symmetric partners:

$$A = \frac{\frac{-b}{a} + 0}{2} = \frac{\frac{-b}{a}}{2} = \frac{-b}{2a}$$

So, the axis of symmetry is $x = \frac{-b}{2a}$.

c) To find the vertex we can use $\frac{-b}{2a}$ as a replacement value for *x* in the original function:

$$y = a\cdot\left(\frac{-b}{2a}\right)^2 + b\cdot\left(\frac{-b}{2a}\right) + c$$

$$y = \frac{b^2}{4a} - \frac{b^2}{2a} + c$$

With common denominators: $$y = \frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a}$$

$$y = \frac{b^2 - 2b^2 + 4ac}{4a}$$

$$y = \frac{-b^2 + 4ac}{4a}$$

So, the vertex is at the point $\left(\frac{-b}{2a}, \frac{-b^2 + 4ac}{4a}\right)$

H. ANOTHER WAY TO DEVELOP THE QUADRATIC FORMULA

You have just seen a general form of the vertex of the quadratic function $y = ax^2 + bx + c$; it is

$$\left(\frac{-b}{2a}, \frac{-b^2 + 4ac}{4a} \right). \tag{1}$$

Using this point, we can rewrite the same function into its vertex form:

$$y = a\cdot\left(x - \frac{-b}{2a} \right)^2 + \frac{-b^2 + 4ac}{4a} \tag{2}$$

This simplifies to:

$$y = a\cdot\left(x + \frac{b}{2a} \right)^2 + \frac{-b^2 + 4ac}{4a} \tag{3}$$

$$y = a\cdot\left(x + \frac{b}{2a} \right)^2 + \frac{-(b^2 - 4ac)}{4a} \tag{4}$$

$$y = a\cdot\left(x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a} \tag{5}$$

At this point, we can find the zeros of the function by letting the right side equal 0. This is the same as letting the standard form equal 0:

so,

$$ax^2 + bx + c = 0 \tag{6}$$

is the same as

$$a\cdot\left(x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a} = 0 \tag{7}$$

Add the constant to each side:

$$a\cdot\left(x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a} \tag{8}$$

Multiply each side by $\frac{1}{a}$:

$$\left(x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{9}$$

Take the square root of each side:

$$\sqrt{\left(x + \frac{b}{2a} \right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \tag{10}$$

$\sqrt{\text{arg}^2} = \left| \text{arg} \right|$

$$\left| x + \frac{b}{2a} \right| = \sqrt{\frac{b^2 - 4ac}{4a^2}} \tag{11}$$

If $\left| \text{arg} \right| = p$, then $\text{arg} = \pm p$:

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \tag{12}$$

Add $-\frac{b}{2a}$ to each side: $$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad (13)$$

Simplify the right side: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (14)$$

This development of the Quadratic Formula was done without once having to complete the square. (Of course, it doesn't matter how you *develop* the quadratic formula, it's still the same formula.)

Exercise: Show that, if $ax^2 + bx + c = 0$, then $x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$

If we use this form of the quadratic formula, are there any restrictions on a, b or c?