Developing the Quadratic Formula, Two Traditional Approaches

THE MOST COMMON TRADITIONAL APPROACH

Completing the square using $(x + d)^2 = x^2 + 2dx + d^2$.

If $ax^2 + bx + c = 0$, $a > 0$*,

we can solve for x by completing the square:

$$ax^2 + bx + c = 0 \tag{1}$$

Divide each side (each term) by a:
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{2}$$

Add $-\frac{c}{a}$ to each side:
$$x^2 + \frac{b}{a}x = -\frac{c}{a} \tag{3}$$

Complete the square by adding the square of half of the linear coefficient to each side:
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \tag{4}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \tag{5}$$

The left side is a perfect square trinomial:
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \tag{6}$$

Simplify the right side of the equation:
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \tag{7}$$

Take the square root of each side:
$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \tag{8}$$

$\sqrt{arg^2} = |\,arg\,|$
$$\left|\,x + \frac{b}{2a}\,\right| = \sqrt{\frac{b^2 - 4ac}{4a^2}} \tag{9}$$

If $|\,arg\,| = p$, then $arg = \pm p$:
$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \tag{10}$$

Add $-\frac{b}{2a}$ to each side:
$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \tag{11}$$

Simplify the right side:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{12}$$

* Having $a > 0$ allows us to write $\sqrt{4a^2} = 2a$ in step (11). There is no loss in generality in saying so, and the quadratic formula is, ultimately, no different if $a < 0$. However, $a \neq 0$ is a requirement.

A LESS COMMON TRADITIONAL APPROACH

Completing the square using $(2ax + b)^2 = 4a^2x^2 + 4abx + b^2$.

If $ax^2 + bx + c = 0,\ a \neq 0$,

we can solve for x by completing the square:

$$ax^2 + bx + c = 0 \qquad (1)$$

Multiply each side by 4a: $$4a^2x^2 + 4abx + 4ac = 0 \qquad (2)$$

Add - 4ac to each side: $$4a^2x^2 + 4abx = -4ac \qquad (3)$$

Complete the square by adding b^2 to each side: $$4a^2x^2 + 4abx + b^2 = b^2 - 4ac \qquad (4)$$

The left side is a perfect square trinomial: $$(2ax + b)^2 = b^2 - 4ac \qquad (5)$$

Take the square root of each side: $$\sqrt{(2ax + b)^2} = \sqrt{b^2 - 4ac} \qquad (6)$$

$\sqrt{\text{arg}^2} = |\text{arg}|$ $$|2ax + b| = \sqrt{b^2 - 4ac} \qquad (7)$$

If $|\text{arg}| = p$, then $\text{arg} = \pm p$: $$2ax + b = \pm\sqrt{b^2 - 4ac} \qquad (8)$$

Add - b to each side: $$2ax = -b \pm \sqrt{b^2 - 4ac} \qquad (9)$$

Divide each side by 2a: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (10)$$

THE QUADRATIC FORMULA

If $ax^2 + bx + c = 0,\ a \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Some prefer to write this as $x = \dfrac{-b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$ in order to reduce student errors oftentimes associated with the application of this formula.