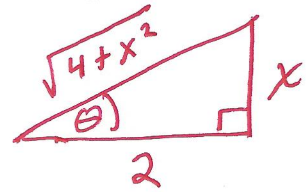
12. $\displaystyle\int \frac{1}{x^2\sqrt{4+x^2}}\,dx$

This is trig substitution; $\sqrt{4+x^2}$ will go on the hypotenuse:



$$= \int \frac{1}{x^2} \cdot \frac{1}{\sqrt{4+x^2}} \cdot dx$$

$$= \int \left(\frac{1}{2}\cot\theta\right)^2 \cdot \frac{1}{2}\cos\theta \cdot 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{4}\cot^2\theta \cdot \frac{1}{2} \cdot 2 \cdot \cos\theta \cdot \frac{1}{\cos^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{\cos^2\theta}{\sin^2\theta} \cdot \frac{\cos\theta}{\cos^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

Let $u = \sin\theta$

$du = \cos\theta\, d\theta$

To get $\frac{1}{x}$, use $\cot\theta$: $\cot\theta = \frac{2}{x}$

$\frac{1}{2}\cot\theta = \frac{1}{x}$

To get $\frac{1}{\sqrt{4+x^2}}$, use $\cos\theta$:

$\cos\theta = \frac{2}{\sqrt{4+x^2}}$

$\frac{1}{2}\cos\theta = \frac{1}{\sqrt{4+x^2}}$

To get dx, use $\tan\theta$:

$\tan\theta = \frac{x}{2}$

$2\tan\theta = x$

$2\sec^2\theta\, d\theta = dx$

$$= \frac{1}{4}\int \frac{1}{u^2}\, du$$

$$= \frac{1}{4}\int u^{-2}\, du$$

$$= \frac{1}{4} \cdot \frac{u^{-1}}{-1}$$

$$= \frac{-1}{4u}$$

resubstitute for u

$$= \frac{-1}{4\sin\theta}$$

$$= -\frac{1}{4}\csc\theta$$

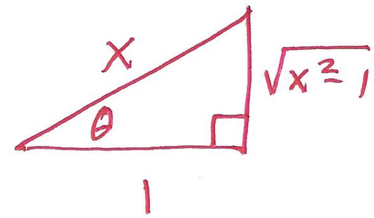
From the triangle: $\csc\theta = \frac{\sqrt{4+x^2}}{x}$

$$= -\frac{1}{4} \cdot \frac{\sqrt{4+x^2}}{x}$$

$$= \boxed{\frac{-\sqrt{4+x^2}}{4x} + C}$$

13. $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx$

This is trig-sub. We can put the $\sqrt{x^2-1}$ on either leg of the Δ:



$$= \int \frac{1}{\sqrt{x^2-1}} \cdot dx$$

To get $\frac{1}{\sqrt{x^2-1}}$, use $\cot\theta$:

$$\cot\theta = \frac{1}{\sqrt{x^2-1}}$$

$$= \int \cot\theta \cdot \sec\theta \cdot \tan\theta \, d\theta$$

$\cot\theta \cdot \tan\theta = 1$ (reciprocals)

To get dx, use $\sec\theta$:

$$\sec\theta = x$$

$$\sec\theta\tan\theta\,d\theta = dx$$

$$= \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\ln\left|x + \sqrt{x^2-1}\right| + C}$$

← We need $\tan\theta$:

$$\tan\theta = \sqrt{x^2-1}$$