[Both techniques are shown here.]

Find the partial fraction decomposition of each.

9. $\frac{2-x}{x^3+x^2} = \frac{2-x}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

Multiply each side by $x^2(x+1)$ to clear all denominators

Technique I: choosing x-values:

$$2 - x = Ax(x+1) + B(x+1) + C \cdot x^2$$

$x=0$: $2 = A \cdot 0 + B \cdot 1 + C \cdot 0 \Rightarrow 2 = B$

$B = 2$

$x=-1$: $3 = A \cdot 0 + B \cdot 0 + C \cdot 1 \Rightarrow 3 = C$

$C = 3$

To find A, let's choose $x = +1$: $1 = A(1)(2) + 2(2) + 3(1)$

$1 = 2A + 4 + 3$

$-6 = 2A \Rightarrow A = -3$

Answer: $\boxed{\frac{-3}{x} + \frac{2}{x^2} + \frac{3}{x+1}}$

Technique II: multiply out and compare like terms

$$2 - x = Ax^2 + Ax + Bx + B + Cx^2$$

$$0x^2 - x + 2 = (A+C)x^2 + (A+B)x + B$$

1 x^2-terms: $0 = A + C \Rightarrow$ .... $C = -A$

2 x-terms: $-1 = A + B \Rightarrow -1 - B = A$

3 Constants: $2 = B$ $\Rightarrow$ $-1 - 2 = A$ $\Rightarrow$ $A = -3$ $\Rightarrow$ $C = 3$

3 4 5

Answer: $\boxed{\frac{-3}{x} + \frac{2}{x^2} + \frac{3}{x+1}}$

10. $\dfrac{3x+7}{x^3-x^2+4x-4}$

first factor the denominator

$(x^3-x^2)+(4x-4)$

$=x^2(x-1)+4(x-1)$

$=(x^2+4)(x-1)$

$$\frac{3x+7}{(x^2+4)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

Second, Clear the denominators

$$3x+7 = A(x^2+4)+(Bx+c)(x-1)$$

Let's use Technique II here, Multiply out and compare like terms.

$$0x^2+3x+7 = Ax^2+4A+Bx^2-Bx+Cx-C$$

$$0x^2+3x+7 = (A+B)x^2+(-B+c)x+(4A-C)$$

x^2- terms: (1) $0 = A+B$

x-terms: (2) $3 = -B+C$

constants: (3) $7 = 4A-C$

None of these combine to give us an equation in one variable. However we can combine equations (1) and (2) to create a new equation in A and C.

(1)+(2) → (4) $3 = A+C$

Now combine this with equation (3) $7 = 4A-C$

$10 = 5A$

$A=2$

(1) $B=-A$, so $B=-2$

(4) $3=2+C$, so $C=1$

Answer: $\dfrac{2}{x-1} + \dfrac{-2x+1}{x^2+4}$

The partial fraction decomposition of $\dfrac{x+4}{x^3+4x}$ is $\dfrac{1}{x} + \dfrac{-x+1}{x^2+4}$ Use this to evaluate the following integral

11. $\displaystyle \int \frac{x+4}{x^3+4x}\,dx = \int \left(\frac{1}{x} + \frac{-x+1}{x^2+4} \right) dx$

$$= \int \left(\frac{1}{x} + \frac{-x}{x^2+4} + \frac{1}{x^2+4} \right) dx$$

$$= \int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$$

$u = x^2+4$

$du = 2x\,dx$

$$= \ln|x| - \frac{1}{2}\int \frac{1}{u}\,du + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \boxed{\ln|x| - \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

It is possible to combine the first two terms: $\ln|x| - \ln|x^2+4|^{1/2} + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

$$= \boxed{\ln\left|\frac{x}{\sqrt{x^2+4}}\right| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

but this isn't necessary.