Determine whether the improper integral converges or diverges. If it converges, find its value.

1. $\displaystyle\int_e^\infty \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{t\to\infty}\int_e^t \frac{1}{x\sqrt{\ln x}}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

x	u
∞	$\ln\infty = \infty$
e	$\ln e = 1$

$$= \lim_{t\to\infty}\int_e^t \frac{1}{\sqrt{\ln x}}\cdot\frac{1}{x}\,dx$$

$$= \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{u}}\,du = \lim_{t\to\infty}\int_1^t u^{-1/2}\,du$$

$$= \lim_{t\to\infty}\frac{2}{1}\cdot u^{1/2}\Big|_1^t = \lim_{t\to\infty}\left(2\sqrt{t} - 2\sqrt{1}\right) = \infty$$

$\boxed{\text{DIVERGES}}$

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2. $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$

$u = e^x$

$du = e^x\,dx$

x	u
∞	$e^\infty = \infty$
0	$e^0 = 1$

$$= \int_0^\infty \frac{1}{(e^x)^2+3}\cdot e^x\,dx$$

$$= \int_1^\infty \frac{1}{u^2+3}\,du$$

This is often written in the form $\displaystyle\int \frac{1}{a^2+u^2}\,du$; the antiderivative is $\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$. In this case, $a = \sqrt{3}$

$$= \lim_{t\to\infty}\int_1^t \frac{1}{3+u^2}\,du = \lim_{t\to\infty}\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\Big|_1^t$$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\cdot\left[\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

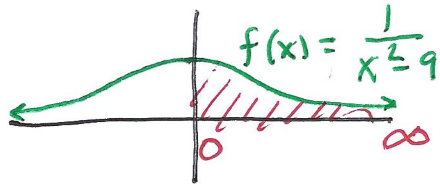
$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$; $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$

$\lim_{t\to\infty}\tan^{-1}(t) = \frac{\pi}{2}$

$$= \frac{\sqrt{3}}{3}\cdot\left[\frac{\pi}{2} - \frac{\pi}{6}\right] = \frac{\sqrt{3}}{3}\left(\frac{3\pi-\pi}{6}\right) = \frac{\sqrt{3}}{3}\cdot\frac{\pi}{3} = \boxed{\frac{\pi\sqrt{3}}{9}}$$

Converges

3. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+9}\,dx$



(even function, symmetric about the y-axis.)

$$= 2\int_0^{\infty} \frac{1}{9+x^2}\,dx$$

this is an inverse-tangent function, just like #2.

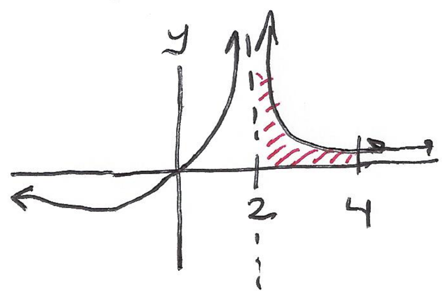
$$= 2\lim_{t\to\infty} \int_0^t \frac{1}{9+x^2}\,dx = 2\lim_{t\to\infty} \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\Big|_0^t$$

$$= \frac{2}{3}\lim_{t\to\infty}\left[\tan^{-1}\left(\frac{t}{3}\right) - \tan^{-1}\left(\frac{0}{3}\right)\right]$$

$$= \frac{2}{3}\left[\frac{\pi}{2} - 0\right] = \frac{2\pi}{6} = \boxed{\frac{\pi}{3}}$$

CONVERGES

4. $\displaystyle\int_{2}^{4} \frac{x}{(4-x^2)^2}\,dx$



$u = 4 - x^2$

$du = -2x\,dx$

x	u
4	-12
2	0

$$= -\frac{1}{2}\int_2^4 \frac{-2x}{(4-x^2)^2}\,dx$$

$$= \cancel{\lim_{t}}\ -\frac{1}{2}\int_0^{-12} \frac{1}{u^2}\,du = +\frac{1}{2}\int_{-12}^{0} \frac{1}{u^2}\,du$$

$$= \lim_{t\to 0} \frac{1}{2}\int_{-12}^{t} \frac{1}{u^2}\,du = \lim_{t\to 0} \frac{1}{2}\cdot\frac{u^{-1}}{-1}\Big|_{-12}^{t}$$

$$= \lim_{t\to 0} \frac{-1}{2u}\Big|_{-12}^{t} = \lim_{t\to 0} \frac{-1}{2t} - \frac{-1}{2(-12)}$$

$$= \infty - \frac{1}{24} = \infty$$

$\boxed{\text{DIVERGES}}$