Factoring the Difference of Squares

We have learned earlier that the *product* (multiplication) of two binomial *conjugates* is the **difference of squares**:

$$(a + b)(a - b) = a^2 - b^2$$

This also means …

> The **difference of squares**, $a^2 - b^2$, factors into a pair of conjugates:
>
> $$a^2 - b^2 = (a - b)(a + b)$$

Note: The binomial conjugate factors in the difference of squares can be written in either order.

> However, the **sum of squares**, $c^2 + d^2$, is *not* factorable; it is ***prime***.

Example 1: Factor each binomial, if possible.

a) $y^2 - 4$ b) $x^2 + 16$ c) $w^2 - 20$

Procedure: Identify each as either the difference of squares or the sum of squares (or neither). The sum of squares is prime, but the difference of squares factors into a product of conjugates.

Answer:

a) $y^2 - 4$
$= (y + 2)(y - 2)$

b) $x^2 + 16$ is *prime*
(The sum of squares is prime.)

c) $w^2 - 20$ is *prime*
(20 is not a perfect square.)

Group Exercise 1 Factor each binomial, if possible.

a) $x^2 - 49$ b) $w^2 + 9$ c) $p^2 + 81$

d) $y^2 - 81$ e) $m^2 - 100$ f) $x^2 - 12$

The difference of squares can come in many forms, as long as each term is a perfect square. Some examples of perfect squares are:

49, which is $(7)^2$ $9y^2$, which is $(3y)^2$ $25w^2$, which is $(5w)^2$

x^4, which is $(x^2)^2$ y^6, which is $(y^3)^2$ and $81w^4$, which is $(9w^2)^2$

Any of these could be a term in the difference of squares, and it will still be factorable. However, the sum of squares is always prime.

Example 2: Factor each binomial, if possible.

a) $9x^2 - 1$ b) $w^6 - 49$ c) $9m^2 + 4p^2$ d) $25x^4 - y^2$

Procedure: Identify the terms as perfect squares, and factor accordingly.

Answer:

a) $9x^2 - 1$ $9x^2 = (3x)^2$ and $1 = (1)^2$

$= (3x + 1)(3x - 1)$

b) $w^6 - 49$ $w^6 = \left(w^3\right)^2$ and $49 = (7)^2$

$= (w^3 - 7)(w^3 + 7)$

c) $9m^2 + 4p^2$ This is the sum of squares.

Prime.

d) $25x^4 - y^2$ $25x^4 = \left(5x^2\right)^2$ and $y^2 = (y)^2$

$= (5x^2 + y)(5x^2 - y)$

Group Exercise 2 Factor each binomial, if possible.

a) $m^2 - 25p^2$ b) $36w^4 - 1$ c) $16y^2 + 49$

d) $64w^2 - 81v^2$ e) $x^4 + 100$ f) $4x^6 - y^2$

FACTORING COMPLETELY

Earlier we learned to recognize and extract the GCF (greatest common factor) of a polynomial. Sometimes, when we factor the GCF from a binomial, the resulting quantity is a difference of squares.

For example, to factor $2x^3 - 18x$ we could first recognize the GCF of $2x$ and extract it from the binomial:

$2x^3 - 18x$ The GCF is $2x$; extract it from each term.

$= 2x(x^2 - 9)$

Notice that the quantify (in the parentheses) is $x^2 - 9$, the difference of squares. This means the polynomial can be factored further. Here is the whole factoring:

$2x^3 - 18x$ The GCF is $2x$; extract it from each term.

$= 2x(x^2 - 9)$ Factor the difference of squares into a product of conjugates.

$= 2x(x - 3)(x + 3)$

In this form, $2x^3 - 18x$ is ***completely*** factored.

Example 5: Factor each polynomial completely.

a) $x^4 - 16x^2$ b) $5w^3 + 45w$ c) $y^4 - 1$

Procedure: First, check for a common monomial factor that can be extracted from each term.
Second, determine whether the quantity factor itself can be factored.
Third, make sure that all factors are included in the final factored form.

Answer:

a) $x^4 - 16x^2$ The GCF is x^2. Extract this from each term.

$= x^2(x^2 - 16)$ The binomial in the quantity, $x^2 - 16$, is the difference of squares and factors into $(x + 4)(x - 4)$.

$= x^2(x + 4)(x - 4)$ Notice that the GCF, x^2, is included in the final factored form.

b) $5w^3 + 45w$ The GCF is $5w$. Extract this from each term.

$= 5w(w^2 + 9)$ The binomial in the quantity, $w^2 + 9$, is the sum of squares and does not factor. This is the final factored form.

c) $y^4 - 1$ The two terms have no common factor, so there is no GCF to extract. However, this is the difference of squares, so we can factor it into the product of two conjugates.

$= (y^2 + 1)(y^2 - 1)$ It would appear as though the factoring is complete except that one of the binomial factors is, itself, the difference of squares. $(y^2 - 1)$ factors into $(y - 1)(y + 1)$.

$= (y^2 + 1)(y - 1)(y + 1)$ This is the final factored form.

Group Exercise 3 Factor each binomial. Be sure to see if a polynomial can be factored more than once.

a) $2x^3 + 18x$ b) $m^4 - 16$ c) $3y^4 - 75y^2$

Focus Exercises

Factor each binomial, if possible.

1. $x^2 - 100$

2. $9m^2 - 64$

3. $49x^2 + 4$

4. $p^6 - 16$

5. $v^4 - 25$

6. $x^2 + 121$

7. $w^3 - 36w$

8. $50x^2 - 18$

9. $5m^3 - 45m$

10. $18x^3 - 8x$

11. $x^4 - 81$

12. $16y^4 - 1$