Solving Quadratic Equations

INTRODUCTION

A **quadratic equation** is an equation in which the degree of the polynomial is 2.

> The **standard form** for a quadratic equation is
>
> $ax^2 + bx + c = 0$ or $0 = ax^2 + bx + c$

Standard form means that one side of this equation is a polynomial in descending order and the other side is 0 (zero).

Group Exercise 1 Write each quadratic equation in standard form.

a) $9 - 6y + 2y^2 = 0$ b) $8 = 3 - 3x + 8x^2$ c) $(x + 3)(x - 2) = 14$

THE ZERO PRODUCT PRINCIPLE

The product (multiplication) of any two non-zero numbers will always be either positive or negative. The only way a product can be zero is if one of the factors is zero, as shown here:

> **The Zero Product Principle**
>
> If the product of two numbers is 0, then one of the numbers must be 0.
>
> **If $A \cdot B = 0$, then either $A = 0$ or $B = 0$.**

The Zero Product Principle can be applied to any two factors that multiply to get 0. For example,

if $\mathbf{(x + 7)(x - 5) = 0}$, then one of the factors must be 0:

either $\mathbf{x + 7 = 0}$ or $\mathbf{x - 5 = 0}$

In other words, either $\mathbf{x = -7}$ or $\mathbf{x = 5}$

It is common to write the two solutions together: $\mathbf{x = -7, 5}$. This is a **solution set**, sometimes written **{-7, 5}**.

Group Exercise 2 Solve the equation. Note: each of these quadratic equations was in standard form but has been factored.

a) $(x + 3)(x - 2) = 0$

b) $0 = (v - 1)(v + 9)$

c) $5x(x + 6) = 0$

d) $(3x - 4)(2x - 5) = 0$

To apply the Zero Product Principle, one side of the equation must be 0 and the other side must be in a factored form. For example, we cannot apply the Zero Product Principle on $x^2 + x - 6 = 0$ until the left side of the equation is factored:

$$x^2 + x - 6 = 0$$

$$(x + 3)(x - 2) = 0$$

Either $\mathbf{x + 3 = 0}$ or $\mathbf{x - 2 = 0}$

Either $\mathbf{x = -3}$ or $\mathbf{x = 2}$

The solution set: $\mathbf{x = -3, 2}$

It is appropriate to verify that these two answers for x, -3 and 2, are actually solutions to the *original* equation. We verify they are *solutions* by "plugging in" each value in the original equations:

Verify $x = -3$

$$x^2 + x - 6 = 0$$

$$(-3)^2 + (-3) - 6 \overset{?}{=} 0$$

$$9 + (-3) - 6 \overset{?}{=} 0$$

$$6 - 6 \overset{?}{=} 0$$

$$0 = 0 \checkmark$$

Verify $x = 2$

$$x^2 + x - 6 = 0$$

$$(2)^2 + (2) - 6 \overset{?}{=} 0$$

$$4 + 2 - 6 \overset{?}{=} 0$$

$$6 - 6 \overset{?}{=} 0$$

$$0 = 0 \checkmark$$

Yes, the solution set is **{-3, 2}**.

Sometimes, verifying that an answer is a solution can be done mentally, as long as it is done carefully.

Group Exercise 3 To verify solutions, it is important to place the answers into the original equation and not the factored form of the equation. Why is this important?

Example 1: Solve each equation.

a) $-2r^2 - 8r = 0$

b) $0 = 2y^2 - 7y + 3$

Procedure: Factor the polynomial and apply the Zero Product Principle. Verify the answers and write the solutions in a solution set.

Answer:

a)

$-2r^2 - 8r = 0$ — Factor the left side. Extract $-2r$.

$-2r(r + 4) = 0$ — Set each factor equal to 0.

$-2r = 0$ or $r + 4 = 0$ — Solve each linear equation.

$r = 0$ or $r = -4$ — A solution may be 0.

$r = \mathbf{0, -4}$ Verify these answers to show they are solutions.

b)

$0 = 2y^2 - 7y + 3$ — Factor the right side. Use the Factor Game.

$0 = (2y - 1)(y - 3)$ — Set each factor equal to 0.

$2y - 1 = 0$ or $y - 3 = 0$ — Solve each linear equation.

$y = \frac{1}{2}$ or $y = 3$ — A solution may be an integer or a fraction.

$y = \mathbf{\frac{1}{2}, 3}$ Verify these answers to show they are solutions.

Notice that, in Example 1b), 0 is on the left side. That is okay. The Zero Product Principle works the same way as long as 0 is on one side or the other.

Group Exercise 4 Solve each equation by factoring.

a) $-5x^2 + 20x = 0$

b) $0 = m^2 + 3m - 28$

> To solve a quadratic equation, the polynomial must first be set equal to 0.

Group Exercise 5 Solve each equation by first putting it in standard form; factor to solve the equation. Verify the answers are solutions to the equation.

a) $y^2 + 2y = 8y - 9$

b) $(3v - 1)(v - 5) = -15$

THE SQUARE ROOT PROPERTY OF EQUATIONS

Consider the equation $x^2 = 25$. There are two ways that we can approach this. Whichever technique we choose, we should get the same solution set.

Technique 1: Solve by factoring

$x^2 = 25$	Add -25 to each side
$x^2 - 25 = 0$	Factor
$(x - 5)(x + 5) = 0$	Solve.
$\boxed{x = 5, -5}$	**Two Solutions.**

Technique 2: Solve by taking the square root of each side:

$$x^2 = 25$$

$$\sqrt{x^2} = \sqrt{25}$$

This appears to have only <u>one solution</u>, $x = 5$. However, we know there are two solutions, and we can represent them as $\boxed{x = \pm 5}$

Technique 2 is an example of ...

The Square Root Property of Equations

If $x^2 = a$	If one side is squared and the other side is a number, we can take the square root of each side to eliminate the exponent.
then $\sqrt{x^2} = \sqrt{a}$	The radicand will have two solutions: one positive and one negative.
and $x = \pm\sqrt{a}$	

Example 2: Solve $(y + 4)^2 = 36$ using the Square Root Property of Equations.

Procedure: Take the square root of each side. The following step must include the ± symbol in front of the evaluated square root.

Answer:

$(y + 4)^2 = 36$	Take the square root of each side.
$\sqrt{(y + 4)^2} = \sqrt{36}$	For the next step, place ± in front of the 6
$y + 4 = \pm 6$	Set $y + 4$ equal to 6 and then to -6.
$y + 4 = 6$ or $y + 4 = -6$	Solve each linear equation.
$y = 2$ or $y = -10$	Combine the solutions.
$y =$ **2, -10**	

Group Exercise 6 Solve each equation using the Square Root Property of Equations. Verify the answers are solutions to the equation.

a) $(x-5)^2 = 16$

b) $(2y+3)^2 = 49$

Focus Exercises

Solve each equation. Verify the answers are solutions to the equation.

1. $3r^2 + 12r = 0$

2. $y^2 - y - 90 = 0$

3. $3p^2 + p = 10$

4. $v(8v + 2) = 15$

5. $w^2 = 6$

6. $(m - 9)^2 = 4$

Supplemental Exercises

Solve each equation. Verify the answers are solutions to the equation.

1. $6x^2 - 54x = 0$

2. $x^2 + 2x - 35 = 0$

3. $2n^2 - 13n + 15 = 0$

4. $4w^2 - 5w - 6 = 0$

5. $9k^2 - 16 = 0$

6. $k^2 - 5k + 2 = -4$

7. $x^2 + 5x - 8 = 28$

8. $w^2 + 2w - 4 = 59$

9. $6w = 5w^2 - 8$

10. $-3x^2 + 4x = -15$

11. $y^2 - y = 18 - 4y$

12. $4x - x^2 = 4$

13. $(x - 3)(x + 7) = -16$

14. $(y + 6)(2y - 3) = -28$

15. $(x - 1)(6x + 5) = x + 3$

16. $m^2 = 81$

17. $(4x + 1)^2 = 25$

18. $(x - 6)^2 = x + 6$