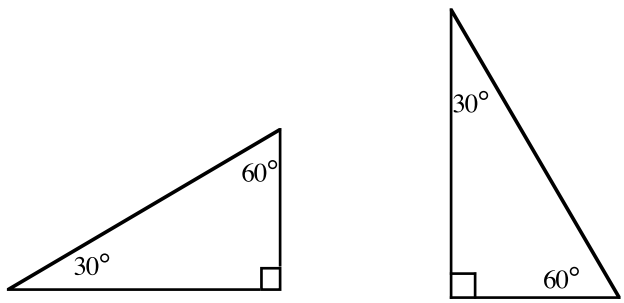
Section 2.1 Special Right Triangles

The 30°-60°-90° Triangle

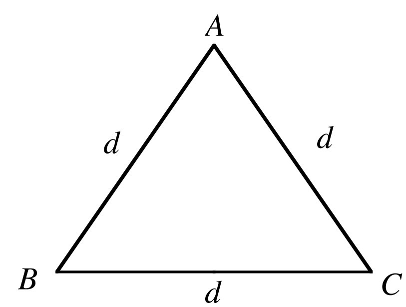
The 30°-60°-90° triangle (or just 30-60-90) is so named because of its angle measures. The lengths of the sides, though, have a very specific pattern to them which we will find useful in the study of trigonometry.



Equilateral = Equiangular

An equilateral (equal side measures) triangle is also known as an ***equiangular*** (equal angle measures) triangle because all three angles have the same measure: 60°. **Why?***

To create a 30-60-90 triangle from an equilateral triangle, follow these steps:



a) In ΔABC, mark $\angle A$, $\angle B$, and $\angle C$ as 60°.

b) In ΔABC, draw an angle bisector from the top vertex, A, to $\overline{BC}$, and label the point of intersection D.

c) Because of symmetry in the equilateral triangle, $\overline{AD}$ is both an altitude and a segment bisector:

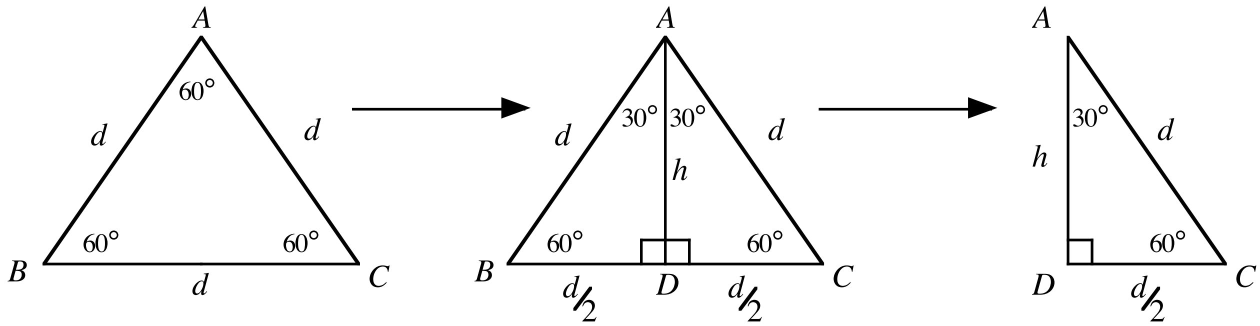
(i) $\angle ADB$ and $\angle ADC$ are right angles, and
(ii) $\overline{AD}$ bisects $\overline{BC}$.

d) Identify the lengths of $\overline{BD}$ and $\overline{CD}$.

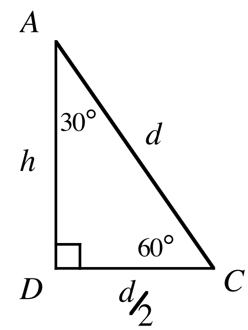
e) At right, draw ΔACD and label side and angle measures; label $\overline{AD}$ with h.

*Because no one side is longer than another, none of the opposite angles can be larger than another, so they must all be the same measure: 180° ÷ 3 = 60°

What you should have discovered on the previous page is that we can establish the 30-60-90 triangle by *bisecting* an equilateral triangle. We can do this on our own if we ever forget the relationships of the sides of a 30-60-90 triangle.



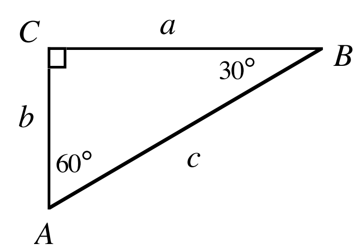
The Ratio of the Shorter Leg to the Hypotenuse



The base, $\overline{BC}$, of the equilateral triangle has the same length, *d*, as the hypotenuse of each of the 30-60-90 triangles. Because the base of the equilateral triangle was bisected, the new bases of each of the two right triangles is half of the original base, $\frac{1}{2}\,d$, or $\frac{d}{2}$.

In a 30-60-90 triangle, the shortest side—which is *half of the hypotenuse*—is always opposite the smallest angle, the 30° angle.

Example 1: In the triangle at right, given the value of *c*, find *b*.

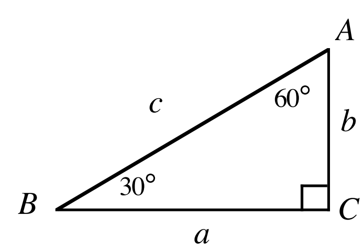


a) $c = 8$ b) $c = 5$ c) $c = 4\sqrt{3}$

Answer: *b* is half of *c*.

a) $b = \frac{8}{2} = 4$ b) $b = \frac{5}{2}$ c) $b = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$

Example 2: In the triangle at right, given the value of *b*, find *c*.



a) $b = 5$ b) $b = \frac{7}{2}$ c) $b = 6\sqrt{3}$

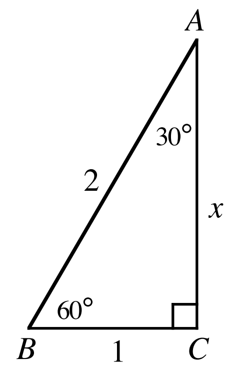
Answer: *c* is twice *b*.

a) $c = 2 \cdot 5 = 10$ b) $c = 2 \cdot \frac{7}{2} = 7$ c) $c = 2 \cdot 6\sqrt{3} = 12\sqrt{3}$

The Ratio of the Longer Leg to the Shorter Leg

The longer leg of the 30-60-90 triangle is opposite the 60° angle. It is the second longest of the three sides, so it must be less than the hypotenuse, d, and more than the shortest leg, $\frac{d}{2}$. We can find the length of this longer leg using the Pythagorean Theorem.

Consider, for example, a 30-60-90 triangle with a hypotenuse of length 2 inches. This means the shorter leg is 1 inch long, and the other leg must have a length, x, between 1 inch and 2 inches. (The same is true if we use feet, meters or miles instead of inches.) This means that $1 < x < 2$.

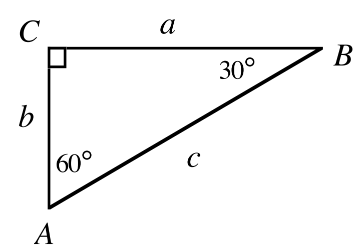


Let's put the Pythagorean Theorem to work:

$1^2 + x^2 = 2^2$	$x = \sqrt{3}$ seems reasonable because:
$1 + x^2 = 4$	$1 < 3 < 4$
$x^2 = 3$	$\sqrt{1} < \sqrt{3} < \sqrt{4}$
$x = \pm\sqrt{3}$	$1 < \sqrt{3} < 2$
$x = \sqrt{3}$ only	By the way, $\sqrt{3} \approx 1.73$

From this example, we see that the longer leg is $\sqrt{3}$ times as long as the shorter leg.

Example 3: In the triangle at right, given the value of ***b***, find ***a***.



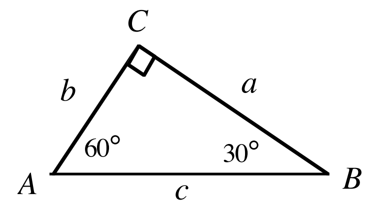
a) $b = 5$ b) $b = \frac{7}{2}$ c) $b = 6\sqrt{3}$

Answer: Multiply the shorter leg by $\sqrt{3}$. $a = b\sqrt{3}$

a) $a = 5\cdot\sqrt{3}$ b) $a = \frac{7}{2}\cdot\sqrt{3}$ or $\frac{7\sqrt{3}}{2}$

c) $a = 6\sqrt{3}\cdot\sqrt{3} = 6\sqrt{9} = 6\cdot 3 = 18$

Example 4: In the triangle at right, given the value of ***a***, find ***b***. Rationalize the denominator if necessary.



a) $a = 4\sqrt{3}$ b) $a = 6$ c) $a = 2$

Answer: Divide the longer leg by $\sqrt{3}$; $b = \dfrac{a}{\sqrt{3}}$

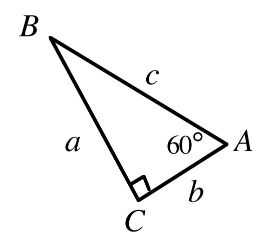
a) $b = \dfrac{4\sqrt{3}}{\sqrt{3}} = 4$

b) $b = \dfrac{6}{\sqrt{3}} = \dfrac{6\sqrt{3}}{3} = 2\sqrt{3}$

c) $b = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

Finding All of the Sides of a 30°-60°-90° Triangle

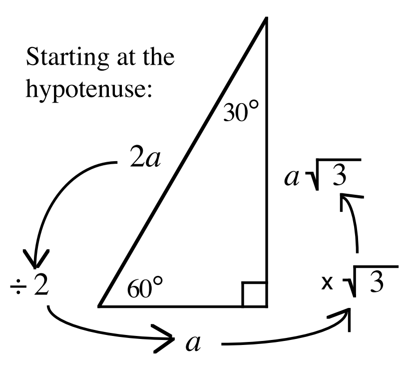
In a 30-60-90 triangle, if you know one side then we can use the relationships described in this section to find the other two sides. The relationships include:

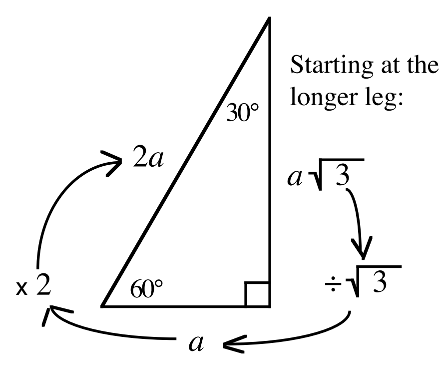


a) A factor of 2 between the hypotenuse and the shortest leg, and

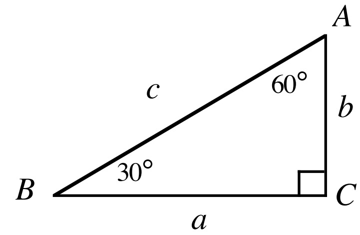
b) A factor of $\sqrt{3}$ between the two legs.

Note: When finding the side length of a 30-60-90- triangle, do not go directly between the longer leg and the hypotenuse; always use the shortest leg as a go-between;





Example 5: In the triangle at right, given the value of a, find b and c.



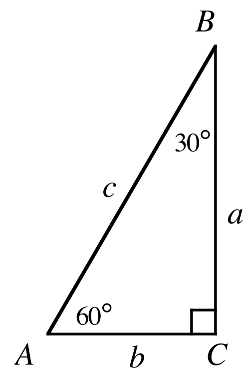
a) $a = 4\sqrt{3}$ b) $a = 6$

Answer: Find the shorter leg, b, first; then find c.

a) $b = \dfrac{4\sqrt{3}}{\sqrt{3}} = 4$ b) $b = \dfrac{6}{\sqrt{3}} = \dfrac{6\sqrt{3}}{3} = 2\sqrt{3}$

$c = 2 \cdot 4 = 8$ $c = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$

Example 6: In the triangle at right, given the value of c, find a and b.



a) $c = 10$ b) $c = 7$

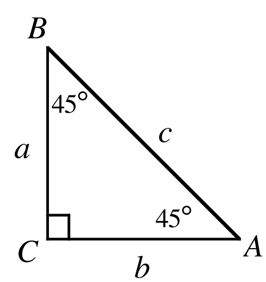
Answer: Find the shorter leg, b, first; then find a.

a) $b = \dfrac{10}{2} = 5$ b) $b = \dfrac{7}{2}$

$a = 5 \cdot \sqrt{3} = 5\sqrt{3}$ $a = \dfrac{7}{2} \cdot \sqrt{3} = \dfrac{7\sqrt{3}}{2}$

The 45°-45°-90° Triangle

Just like the 30°-60°-90° triangle, the 45°-45°-90° is very important in the study of trigonometry. And, no matter the size, every 45-45-90 triangle is *similar** to every other 45-45-90 triangle.



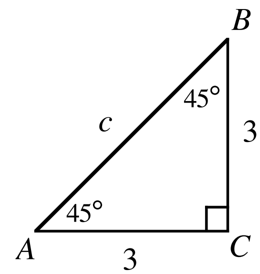
The 45-45-90 triangle is not only a right triangle, it is an *isosceles* triangle.

Because the triangle is isosceles, the legs are congruent to each other. This means that knowing the length of one of the legs automatically tells you the length of the other leg. The hypotenuse can be found by applying the Pythagorean Theorem.

*Similar triangles have the same angle measures and the same proportion between consecutive sides. In other words, no matter the size of a 45-45-90 triangle, the proportion between any pair of sides is the same within each 45-45-90 triangle.

The Ratio of the Hypotenuse to a Leg

To demonstrate the relationship between the lengths of the hypotenuse and one of the legs, let's arbitrarily choose to have the length of one of the legs be 3 inches. Of course, the other leg is also 3 inches, so let's find the hypotenuse.



$$3^2 + 3^2 = c^2$$

$$9 + 9 = c^2$$

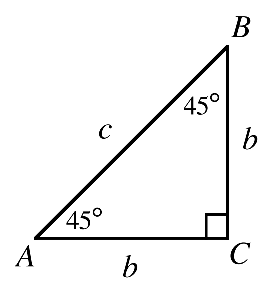
$$18 = c^2$$

$$\pm\sqrt{18} = c$$

$$c = 3\cdot\sqrt{2}$$

This suggests that the hypotenuse is $\sqrt{2}$ times the length of a leg, either leg. This is true for every 45-45-90 triangle.

To demonstrate this consistency, let's choose the length of each leg to be much more arbitrary and let its value just be b. This leads to the following:



$$b^2 + b^2 = c^2$$

$$2b^2 = c^2$$

$$\pm\sqrt{2b^2} = c$$

$$c = \sqrt{2\cdot b^2} = \sqrt{2} \cdot b \text{ or } b\sqrt{2}$$

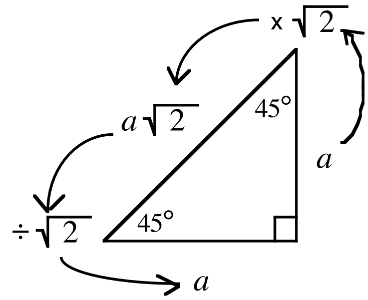
Again, the hypotenuse is $\sqrt{2}$ times as long as either leg. This is consistent with what we found before.

Finding All of the Sides of a 45°-45°-90° Triangle

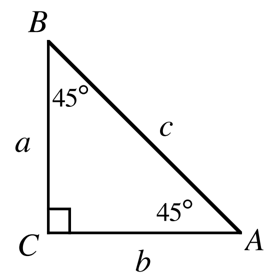
In a 45-45-90 triangle, if you know one side then we can use the relationships described above to find the other two sides. The relationships include:

a) The two legs are congruent, and

b) A factor of $\sqrt{2}$ between the hypotenuse and either leg.



Example 7: In the triangle at right, given the value of ***a***, find ***b*** and ***c***.



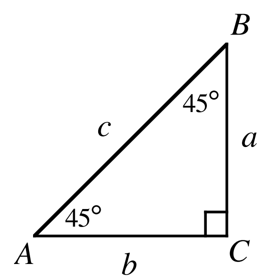
a) $a = 4$ b) $a = 5\sqrt{2}$

Answer: Find the other leg, ***b***; then find ***c***.

a) $b = 4$ b) $b = 5\sqrt{2}$

$c = 4 \cdot \sqrt{2} = 4\sqrt{2}$ $c = 5\sqrt{2} \cdot \sqrt{2} = 5 \cdot 2 = 10$

Example 8: In the triangle at right, given the value of ***c***, find ***a*** and ***b***.



a) $c = 3\sqrt{2}$ b) $c = 8$

Answer: Find either leg, let's say *b*, first; then find the other leg, *a*.

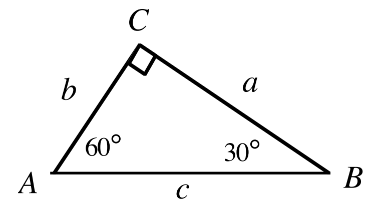
a) $b = \dfrac{3\sqrt{2}}{\sqrt{2}} = 3$ b) $b = \dfrac{8}{\sqrt{2}} = \dfrac{8\cdot\sqrt{2}}{2} = 4\sqrt{2}$

$a = 3$ $a = 4\sqrt{2}$

Section 2.1 Focus Exercises

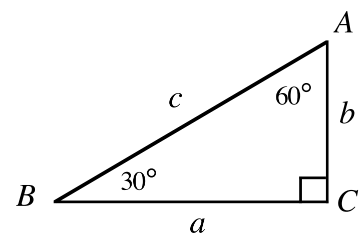
1. Given the length of b, find the lengths of a and c.

a) $b = 5$ b) $b = \frac{9}{2}$ c) $b = 7\sqrt{3}$



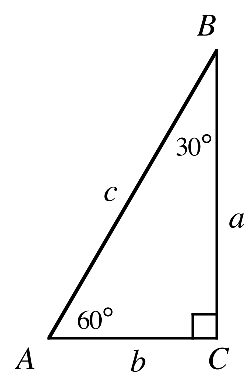
2. Given the length of c, find the lengths of a and b.

a) $c = 5$ b) $c = \frac{9}{2}$ c) $c = 7\sqrt{3}$



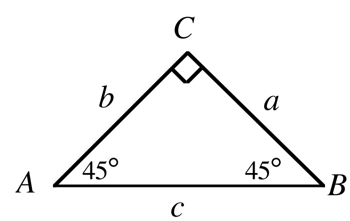
3. Given the length of a, find the lengths of b and c.

a) $a = 10\sqrt{3}$ b) $a = 12$ c) $a = 5$



4. Given the length of a or b, find the lengths of the other two sides.

a) $a = 5$ b) $a = 9\sqrt{2}$ c) $b = 4\sqrt{3}$



5. Given the length of c, find the lengths of a and b.

a) $c = 6\sqrt{2}$ b) $c = 14$ c) $c = 11$