Section 2.1 Special Right Triangles

The 30°-60°-90° Triangle

The 30°-60°-90° triangle (or just 30-60-90) is so named because of its angle measures. The lengths of the sides, though, have a very specific pattern to them which we will find useful in the study of trigonometry.

Equilateral = Equiangular

An equilateral (equal-side-measure) triangle is also known as an equiangular (equal-angle-measure) triangle because all three angles have the same measure: 60°. Why?*

To create a 30-60-90 triangle from an equilateral triangle, follow these steps on ΔABC:

1. In ΔABC, mark ∠A, ∠B, and ∠C as 60°.
2. In ΔABC, draw an angle bisector from the top vertex, A, to BC, and label the point of intersection D.
3. Because of symmetry in the equilateral triangle, AD is both an altitude and a segment bisector:
   a) ∠ADB and ∠ADC are right angles, and
   b) AD bisects BC.
4. Identify the lengths of BD and CD.
5. At right, draw ΔACD and label the side and angle measures; label AD with h.

*Because no one side is longer than another, none of the opposite angles can be larger than another, so they must all be the same measure: 180° ÷ 3 = 60°
What you should have discovered on the previous page is that we can establish the 30-60-90 triangle by bisecting an equilateral triangle. We can do this on our own if we ever forget the relationships of the sides of a 30-60-90 triangle.

**The Ratio of the Shorter Leg to the Hypotenuse**

The base, \(BC\), of the equilateral triangle has the same length, \(d\), as the hypotenuse of each of the 30-60-90 triangles. Because the base of the equilateral triangle was bisected, the new bases of each of the two right triangles is half of the original base, \(\frac{1}{2} \cdot d\), or \(\frac{d}{2}\).

In a 30-60-90 triangle, the shortest side—which is half of the hypotenuse—is always opposite the smallest angle, the 30° angle.

**Example 1:** In the triangle at right, given the value of \(c\), find \(b\).

a) \(c = 8\)  
   b) \(c = 5\)  
   c) \(c = 4\sqrt{3}\)

**Answer:** \(b\) is half of \(c\).

a) \(b = \frac{8}{2} = 4\)  
   b) \(b = \frac{5}{2}\)  
   c) \(b = \frac{4\sqrt{3}}{2} = 2\sqrt{3}\)

**Example 2:** In the triangle at right, given the value of \(b\), find \(c\).

a) \(b = 5\)  
   b) \(b = \frac{7}{2}\)  
   c) \(b = 6\sqrt{3}\)

**Answer:** \(c\) is twice \(b\): \(c = 2 \cdot b\)

a) \(c = 2 \cdot 5 = 10\)  
   b) \(c = 2 \cdot \frac{7}{2} = 7\)  
   c) \(c = 2 \cdot 6\sqrt{3} = 12\sqrt{3}\)
The Ratio of the Longer Leg to the Shorter Leg

The longer leg of the 30-60-90 triangle is opposite the 60° angle. It is the second longest of the three sides, so it must be less than the hypotenuse, \( d \), and more than the shortest leg, \( \frac{d}{2} \). We can find the length of this longer leg using the Pythagorean Theorem.

Consider, for example, a 30-60-90 triangle with a hypotenuse of length 2 inches. This means the shorter leg is 1 inch long, and the other leg must have a length, \( x \), between 1 inch and 2 inches. (The same is true if we use feet, meters or miles instead of inches.) This means that \( 1 < x < 2 \).

Let’s put the Pythagorean Theorem to work:

\[
\begin{align*}
1^2 + x^2 &= 2^2 \\
1 + x^2 &= 4 \\
x^2 &= 3 \\
x &= \pm \sqrt{3} \\
\end{align*}
\]

\( x = \sqrt{3} \) seems reasonable because:

\[
\begin{align*}
1 < 3 < 4 \\
\sqrt{1} < \sqrt{3} < \sqrt{4} \\
1 < \sqrt{3} < 2 \\
\end{align*}
\]

\( x = \sqrt{3} \) only

By the way, \( \sqrt{3} \approx 1.73 \)

From this example, we see that the longer leg is \( \sqrt{3} \) times as long as the shorter leg.

Example 3:  In the triangle at right, given the value of \( b \), find \( a \).

\[
\begin{align*}
a) \quad b &= 5 \\
b) \quad b &= \frac{7}{2} \\
c) \quad b &= 6\sqrt{3}
\end{align*}
\]

Answer:  Multiply the shorter leg by \( \sqrt{3} \). \( a = b\sqrt{3} \)

\[
\begin{align*}
a) \quad a &= 5\sqrt{3} \\
b) \quad a &= \frac{7}{2} \sqrt{3} \quad \text{or} \quad \frac{7\sqrt{3}}{2} \\
c) \quad a &= 6\sqrt{3} \cdot \sqrt{3} = 6\sqrt{9} = 6 \cdot 3 = 18
\end{align*}
\]
Example 4: In the triangle at right, given the value of \( a \), find \( b \).
Rationalize the denominator if necessary.

a) \( a = 4\sqrt{3} \)  

b) \( a = 6 \)  

c) \( a = 2 \)

**Answer:** Divide the longer leg by \( \sqrt{3} \); \( b = \frac{a}{\sqrt{3}} \)

a) \( b = \frac{4\sqrt{3}}{\sqrt{3}} = 4 \)  

b) \( b = \frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3} \)

c) \( b = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \)

**Finding All of the Sides of a 30°-60°-90° Triangle**

In a 30-60-90 triangle, if you know one side then we can use the relationships described in this section to find the other two sides. The relationships include:

a) A factor of 2 between the hypotenuse and the shortest leg, and

b) A factor of \( \sqrt{3} \) between the two legs.

**Note:** When finding the side length of a 30-60-90- triangle, do not go directly between the longer leg and the hypotenuse; always use the shortest leg as a go-between;
Example 5: In the triangle at right, given the value of \(c\), find \(a\) and \(b\).

a) \(c = 10\)  

\[ \frac{10}{2} = 5 \]

\[ a = 5 \cdot \sqrt{3} = 5\sqrt{3} \]

b) \(c = 7\)

\[ \frac{7}{2} = \frac{7}{2} \cdot \sqrt{3} = \frac{7\sqrt{3}}{2} \]

Answer: Find the shorter leg, \(b\), first; then find \(a\).

Example 6: In the triangle at right, given the value of \(a\), find \(b\) and \(c\).

a) \(a = 4\sqrt{3}\)

\[ \frac{4\sqrt{3}}{\sqrt{3}} = 4 \]

\[ c = 2 \cdot 4 = 8 \]

b) \(a = 6\)

\[ \frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3} \]

\[ c = 2 \cdot 2\sqrt{3} = 4\sqrt{3} \]

Answer: Find the shorter leg, \(b\), first; then find \(c\).

The 45°-45°-90° Triangle

Just like the 30°-60°-90° triangle, the 45°-45°-90° is very important in the study of trigonometry. And, no matter the size, every 45-45-90 triangle is similar\(^*\) to every other 45-45-90 triangle.

The 45-45-90 triangle is not only a right triangle, it is an isosceles triangle.

Because the triangle is isosceles, the legs are congruent to each other. This means that knowing the length of one of the legs automatically tells you the length of the other leg. The hypotenuse can be found by applying the Pythagorean Theorem.

\(^*\)Similar triangles have the same angle measures and the same proportion between consecutive sides. In other words, no matter the size of a 45-45-90 triangle, the proportion between any pair of sides is the same within each 45-45-90 triangle.
The Ratio of the Hypotenuse to a Leg

To demonstrate the relationship between the lengths of the hypotenuse and one of the legs, let’s arbitrarily choose to have the length of one of the legs be 3 inches. Of course, the other leg is also 3 inches, so let’s find the hypotenuse.

\[ 3^2 + 3^2 = c^2 \]
\[ 9 + 9 = c^2 \]
\[ 18 = c^2 \]
\[ \pm\sqrt{18} = c \]
\[ c = 3\sqrt{2} \]

This suggests that the hypotenuse is \( \sqrt{2} \) times the length of a leg, either leg. This is true for every 45-45-90 triangle.

To demonstrate this consistency, let’s choose the length of each leg to be much more arbitrary and let its value just be \( b \). This leads to the following:

\[ b^2 + b^2 = c^2 \]
\[ 2b^2 = c^2 \]
\[ \pm\sqrt{2b^2} = c \]
\[ c = \sqrt{2b^2} = \sqrt{2} \cdot b \text{ or } b\sqrt{2} \]

Again, the hypotenuse is \( \sqrt{2} \) times as long as either leg. This is consistent with what we found before.
Finding All of the Sides of a 45°-45°-90° Triangle

In a 45-45-90 triangle, if you know one side then we can use the relationships described above to find the other two sides. The relationships include:

a) The two legs are congruent, and

b) A factor of \( \sqrt{2} \) between the hypotenuse and either leg.

Example 7: In the triangle at right, given the value of \( a \), find \( b \) and \( c \).

a) \( a = 4 \) 

b) \( a = 5\sqrt{2} \)

Answer: Find the other leg, \( b \); then find \( c \).

a) \( b = 4 \) 

b) \( b = 5\sqrt{2} \)

\[
c = 4 \cdot \sqrt{2} = 4\sqrt{2} \quad \text{and} \quad c = 5\sqrt{2} \cdot \sqrt{2} = 5 \cdot 2 = 10
\]

Example 8: In the triangle at right, given the value of \( c \), find \( a \) and \( b \).

a) \( c = 3\sqrt{2} \) 

b) \( c = 8 \)

Answer: Find either leg, let’s say \( b \), first; then find the other leg, \( a \).

a) \( b = \frac{3\sqrt{2}}{\sqrt{2}} = 3 \) 

b) \( b = \frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2} \)

\[
a = 3 \quad \text{and} \quad a = 4\sqrt{2}
\]
Sec. 2.1 Special Right Triangles

Separating Triangles

Sometimes a diagram will include more than one triangle, as demonstrated in Example 9. One approach to finding the sides of the triangle to re-draw the diagram as two separate triangles. When doing so, it is important to label the triangles correctly. (We saw the separation of triangles at the beginning of the section when we split apart the two 30-60-90 triangles.)

Example 9: Given the length of one side, find the other four side measures in these triangles. Simplify completely.

\[ x = 8 \]

Procedure: First separate the two triangles and orient them in a familiar way, as shown below. Label each triangles according to the original diagram.

Note: The hypotenuse of the 30-60-90 triangle is labeled \( x \), and the hypotenuse of the 45-45-90 triangle is labeled \( m \).

Because we are given the value of \( x \), we must start in the 30-60-90 triangle and find both \( y \) and \( m \), in that order:

\[ y = \frac{1}{2} \cdot x \quad \text{and} \quad m = y \cdot \sqrt{3} \]

Once we know the value of \( m \), we can find the values of \( p \) and \( h \).

\[ h = \frac{m}{\sqrt{2}} \quad \text{and} \quad p = h. \]

Answer: \[ y = \frac{1}{2} \cdot 8 = 4 \quad \Rightarrow \quad \text{so} \quad m = 4\sqrt{3} \quad \ldots \quad \text{and} \]

\[ h = \frac{4\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{6}}{2} = 2\sqrt{6} \quad \Rightarrow \quad \text{so} \quad p = 2\sqrt{6} \]
Section 2.1 Focus Exercises

For #1-12, each 30-60-90 triangle is labeled separately, so $\angle B$ could be the smaller angle or the larger angle; in each case, $\angle C$ is the right angle.

Given the length of $b$, find the lengths of $a$ and $c$.

1. $b = \frac{9}{2}$
2. $b = 7\sqrt{3}$
3. $b = 4\sqrt{2}$
4. $b = \frac{5\sqrt{3}}{3}$

Given the length of $c$, find the lengths of $a$ and $b$.

5. $c = 12$
6. $c = 3$
7. $c = 10\sqrt{3}$
8. $c = \frac{\sqrt{3}}{2}$
Given the length of $b$, find the lengths of $a$ and $c$.

9. $b = 9\sqrt{3}$

10. $b = 12$

11. $b = 3\sqrt{2}$

12. $b = 2\sqrt{5}$

For #13-20, in each 45-45-90 triangle, $\angle C$ is the right angle.

Given the length of $a$, find the lengths of $b$ and $c$.

13. $a = 5$

14. $a = 9\sqrt{2}$

15. $a = 4\sqrt{3}$

16. $a = 7\sqrt{6}$
Given the length of $c$, find the lengths of $a$ and $b$.

17. $c = 6\sqrt{2}$
18. $c = 14$

19. $c = 11$
20. $c = 3\sqrt{5}$

Given one of the values of $h$, $m$, $p$, $x$, and $y$, find the other four values shown in the diagram. Simplify completely.

Use the diagram at right for #21-24.

21. $m = 8$

22. $y = 6$
23. $p = 20$
24. $h = 9$
Use the diagram at right for #25-28.

25. \( m = 2\sqrt{3} \)

26. \( p = 14 \)  
27. \( y = 6 \)  
28. \( x = 15 \)

Use the diagram at right for #29-32.

29. \( p = 6 \)

30. \( y = \frac{\sqrt{3}}{2} \)  
31. \( m = 12 \)  
32. \( x = 1 \)