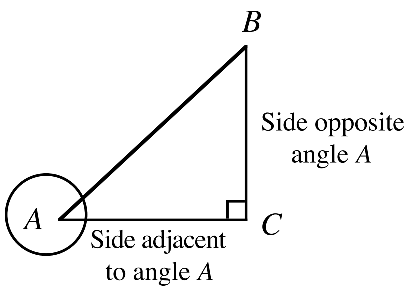
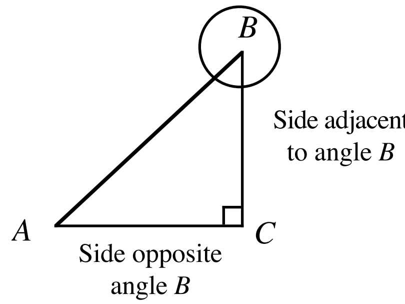
Section 2.2 Trigonometry: The Triangle Identities

ADJACENT AND OPPOSITE SIDES

The study of *triangle* trigonometry is centered around the acute angles in a right triangle. In a right triangle, we call the leg that forms an acute angle with the hypotenuse the ***adjacent side***, or ***side adjacent***, to that angle; the other leg is the ***opposite side***, or ***side opposite***.

For example, in the triangles below, side $\overline{AC}$ is *adjacent* to $\angle A$ and side $\overline{BC}$ is *opposite* $\angle A$. Also, in reference to angle B, side $\overline{BC}$ is adjacent to $\angle B$ and side $\overline{AC}$ is opposite $\angle B$.





THE SINE AND COSINE FUNCTIONS

In a function, $f(x)$, the value in parentheses is called the ***argument*** of the function. For example, if $f(x) = 2x - 5$, then x is the argument. If we replace x with 3 and get $f(3) = 2(3) - 5$, then 3 is the argument.

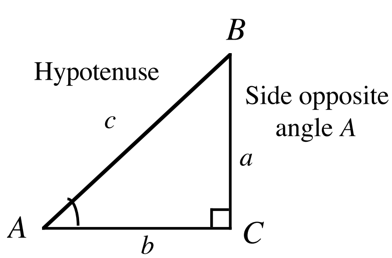
In triangle trigonometry, ***sine*** and ***cosine*** are functions, and the arguments are the acute angle measures in a right triangle*. For example, in *the sine of 70°*, written sin(70°), 70° is the argument; likewise, in *the cosine of* $\angle B$, written $\cos(B)$, the argument is B.

In right triangle ABC:

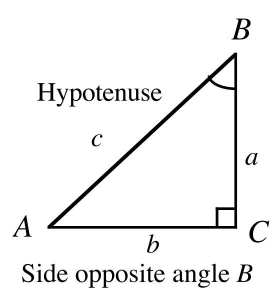
The *sine* of an acute angle is the ratio of the <u>side opposite the angle</u> to <u>the hypotenuse</u>.

As a ratio,

$$\sin(A) = \frac{\text{the side opposite } A}{\text{the hypotenuse}} = \frac{\text{opp } A}{\text{hyp}}$$



$$\sin(B) = \frac{\text{the side opposite } B}{\text{the hypotenuse}} = \frac{\text{opp } B}{\text{hyp}}$$

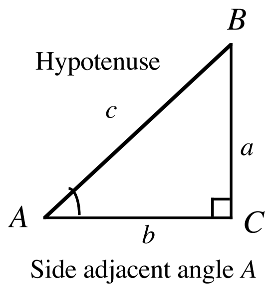


*As with any function, trigonometric functions must have an argument; e.g., cos without an argument is meaningless.

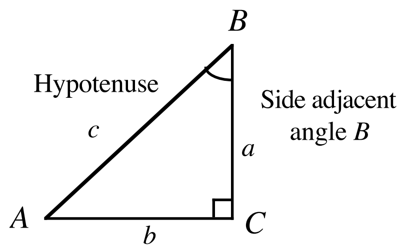
The ***cosine*** of an acute angle is the ratio of the side adjacent to the angle to the hypotenuse.

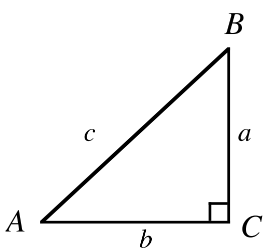
As a ratio,

$$\cos(A) = \frac{\text{the side adjacent } A}{\text{the hypotenuse}} = \frac{\text{adj } A}{\text{hyp}}$$



$$\cos(B) = \frac{\text{the side adjacent } B}{\text{the hypotenuse}} = \frac{\text{adj } B}{\text{hyp}}$$





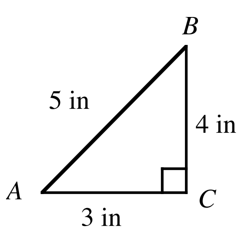
To this point we have the following:

$$\sin(A) = \frac{\text{opp } A}{\text{hyp}} = \frac{a}{c} \qquad \sin(B) = \frac{\text{opp } B}{\text{hyp}} = \frac{b}{c}$$

$$\cos(A) = \frac{\text{adj } A}{\text{hyp}} = \frac{b}{c} \qquad \cos(B) = \frac{\text{adj } B}{\text{hyp}} = \frac{a}{c}$$

Example 1: Given each triangle, find the sine and cosine of angles A and B. Rationalize the denominator if necessary.

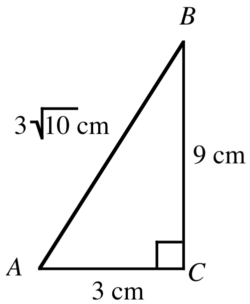
a)



$$\sin(A) = \frac{4 \text{ in}}{5 \text{ in}} = \frac{4}{5} \qquad \sin(B) = \frac{3 \text{ in}}{5 \text{ in}} = \frac{3}{5}$$

$$\cos(A) = \frac{3 \text{ in}}{5 \text{ in}} = \frac{3}{5} \qquad \cos(B) = \frac{4 \text{ in}}{5 \text{ in}} = \frac{4}{5}$$

b)



$$\sin(A) = \frac{9 \text{ cm}}{3\sqrt{10} \text{ cm}} = \frac{3}{\sqrt{10}} = \frac{3}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

$$\cos(A) = \frac{3 \text{ cm}}{3\sqrt{10} \text{ cm}} = \frac{1}{\sqrt{10}} = \frac{1}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$\sin(B) = \frac{3 \text{ cm}}{3\sqrt{10} \text{ cm}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$\cos(B) = \frac{9 \text{ cm}}{3\sqrt{10} \text{ cm}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

Important note: In the examples above, the argument of each function is an angle (*A* or *B*), but what is the result (output) of the function?

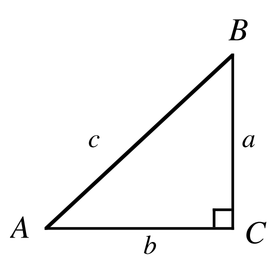
Each result is a ratio in which the units of measure divide out. So, each output is a real number independent of any units.

From this point on, the side measures of each triangle will not include units.

THE TANGENT FUNCTION

A third trigonometric function, the ***tangent***, is a ratio comparing the two legs of the triangle.

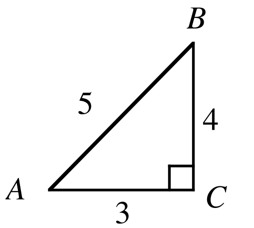
As a ratio, $\tan(A) = \frac{\text{the side opposite } A}{\text{the side adjacent } A} = \frac{\text{opp } A}{\text{adj } A}$



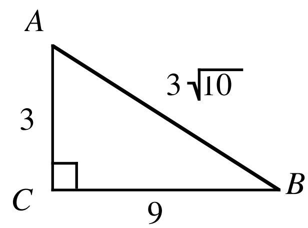
$\tan(A) = \frac{\text{opp } A}{\text{adj } A} = \frac{a}{b}$ $\qquad$ $\tan(B) = \frac{\text{opp } B}{\text{adj } B} = \frac{b}{a}$

Example 2: Given the right triangle below, find the tangent of angles A and B.

a)



b)



Answer: a) $\tan(A) = \frac{4}{3}$ b) $\tan(A) = \frac{9}{3} = 3$

$\tan(B) = \frac{3}{4}$ $\tan(B) = \frac{3}{9} = \frac{1}{3}$

It is important to memorize the sine, cosine, and tangent ratios.

SOH-CAH-TOA

A helpful aid in remembering the triangle trig relationships is "Soh-Cah-Toa," which means:

Soh: **S**ine is $\frac{\textbf{O}\text{pposite}}{\textbf{H}\text{ypotenuse}}$ *Cah:* **C**osine is $\frac{\textbf{A}\text{djacent}}{\textbf{H}\text{ypotenuse}}$ *Toa:* **T**angent is $\frac{\textbf{O}\text{pposite}}{\textbf{A}\text{djacent}}$

The Reciprocal Functions

Each of the three major trig functions has a reciprocal. They are listed below.

Note: $\frac{1}{b}$ means "the reciprocal of *b*." i.e., the reciprocal of *b* can be represented by $\frac{1}{b}$.

THE SECANT FUNCTION

The ***secant*** of an angle, *A*, abbreviated sec(*A*), is the reciprocal of the cosine function:

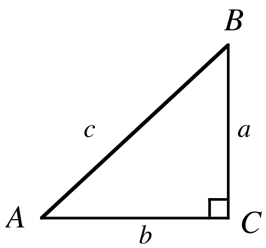
$$\mathbf{sec}(A) = \frac{1}{\mathbf{cos}(A)} \quad \text{and} \quad \mathbf{cos}(A) = \frac{1}{\mathbf{sec}(A)}$$ ⇐ *Memorize these!*

For example, if $\cos(A) = \frac{2}{7}$, then $\sec(A) = \frac{7}{2}$; also, if $\sec(A) = 4$, then $\cos(A) = \frac{1}{4}$.

We can also relate the secant function to the right triangle:

$$\sec(A) = \frac{\text{hypotenuse}}{\text{the side adjacent } A} = \frac{\text{hyp}}{\text{adj } A}$$

However, it is common to find sec(A) only after first using the triangle to find cos(A). In other words, we rarely use the right triangle to find sec(A).

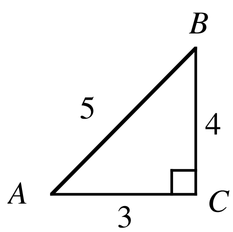


$$\cos(A) = \frac{\text{adj } A}{\text{hyp}} = \frac{b}{c}, \text{ so } \sec(A) = \frac{c}{b}$$

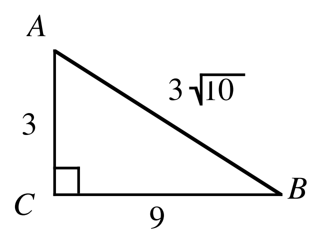
$$\cos(B) = \frac{\text{adj } B}{\text{hyp}} = \frac{a}{c}, \text{ so } \sec(B) = \frac{c}{a}$$

Example 3: Given the triangle below, find the secant of angles A and B. Rationalize the denominator if necessary.

a)



b)



Answer: For each, first find the cosine of the angle, and then find the secant as the reciprocal of the cosine. In part b), we decide whether to rationalize the denominator only after we see whether the secant function requires it.

a) $\cos(A) = \frac{3}{5}$, so $\sec(A) = \frac{5}{3}$; $\cos(B) = \frac{4}{5}$, so $\sec(B) = \frac{5}{4}$

b) $\cos(A) = \frac{3}{3\sqrt{10}} = \frac{1}{\sqrt{10}}$ (don't rationalize yet) ; $\sec(A) = \frac{\sqrt{10}}{1} = \sqrt{10}$

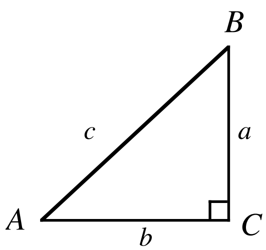
$\cos(B) = \frac{9}{3\sqrt{10}} = \frac{3}{\sqrt{10}}$ (don't rationalize yet) ; $\sec(B) = \frac{\sqrt{10}}{3}$

THE COSECANT FUNCTION

The cosecant of an angle, A, abbreviated csc(A), is the reciprocal of the sine function:

$$\mathbf{csc}(A) = \frac{\mathbf{1}}{\mathbf{sin}(A)} \quad \text{and} \quad \mathbf{sin}(A) = \frac{\mathbf{1}}{\mathbf{csc}(A)}$$ ⇐ *Memorize these!*

As with the secant function, it is common to find sec(A) only after first using the triangle to find sin(A). In other words, we rarely use the right triangle to find csc(A).

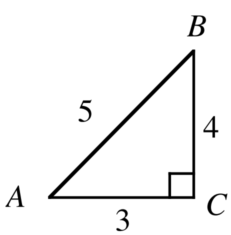


$$\sin(A) = \frac{\text{opp } A}{\text{hyp}} = \frac{a}{c}, \text{ so } \csc(A) = \frac{c}{a}$$

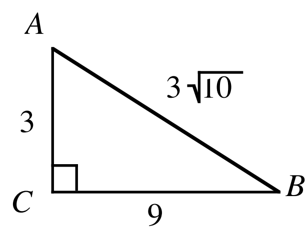
$$\sin(B) = \frac{\text{opp } B}{\text{hyp}} = \frac{b}{c}, \text{ so } \csc(B) = \frac{c}{b}$$

Example 4: Given the triangle below, find the cosecant of angles A and B. Rationalize the denominator if necessary.

a)



b)



Answer: For each, first find the sine of the angle, and then find the cosecant as the reciprocal of the sine. In part b), we decide whether to rationalize the denominator only after we see whether the cosecant function requires it.

a) $\sin(A) = \frac{4}{5}$, so $\csc(A) = \frac{5}{4}$; $\sin(B) = \frac{3}{5}$, so $\csc(B) = \frac{5}{3}$

b) $\sin(A) = \frac{9}{3\sqrt{10}} = \frac{3}{\sqrt{10}}$ (don't rationalize yet) ; $\csc(A) = \frac{\sqrt{10}}{3}$

$\sin(B) = \frac{3}{3\sqrt{10}} = \frac{1}{\sqrt{10}}$ (don't rationalize yet) ; $\csc(B) = \frac{\sqrt{10}}{1} = \sqrt{10}$

THE COTANGENT FUNCTION

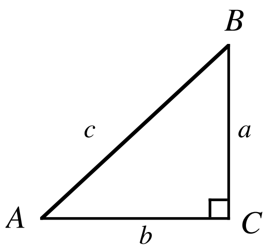
The cotangent of an angle A, abbreviated cot(A), is the reciprocal of the tangent function:

$$\mathbf{cot(A) = \frac{1}{tan(A)}} \quad \text{and} \quad \mathbf{tan(\mathit{A}) = \frac{1}{cot(\mathit{A})}} \quad \Leftarrow \textit{Memorize these!}$$

Relating the cotangent function to the right triangle, we get:

$$\cot(A) = \frac{\text{the side adjacent } A}{\text{the side opposite } A} = \frac{\text{adj } A}{\text{opp } A}$$

Note: In triangle trigonometry, it is common to first find the tangent of an angle before finding the cotangent, but it is also common to know the triangle relationship defined above.

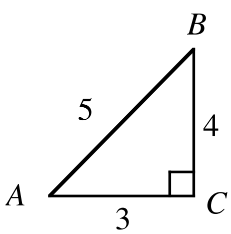


$$\cot(A) = \frac{\text{adj A}}{\text{opp A}} = \frac{b}{a} \qquad \cot(B) = \frac{\text{adj B}}{\text{opp B}} = \frac{a}{b}$$

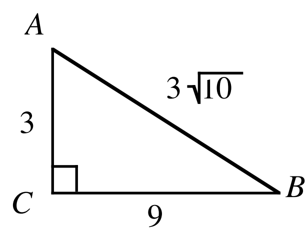
$$\left(\tan(A) = \frac{a}{b} \text{ and } \tan(B) = \frac{b}{a}\right)$$

Example 5: Given the triangle below, find the cotangent of angles *A* and *B*. Rationalize the denominator if necessary.

a)



b)



Answer: a) $\cot(A) = \frac{3}{4}$ b) $\cot(A) = \frac{3}{9} = \frac{1}{3}$

$\cot(B) = \frac{4}{3}$ $\cot(B) = \frac{9}{3} = 3$

CO-FUNCTIONS

You've probably noticed some interesting connections between some of the trigonometric functions. For example, every function has a co-function:

sine and cosine are co-functions

tangent and cotangent are co-functions

secant and cosecant are co-functions

That's probably not a surprise. What might be a surprise, though, is that secant is the reciprocal of ***co***sine, whereas ***co***secant is the reciprocal of sine. Please learn this well—memorize these relationships—as they are *very* important to the success of learning trigonometry.

The connection between a pair of co-functions is *complementary angles*. Specifically, the sine of an angle is the cosine of its complement:

$$\sin(A) = \cos(90° - A)$$

For example, 20° and 70° are complementary angles so it must be that

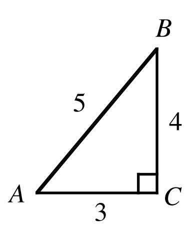
$\sin(20°) = \cos(70°)$ and $\sin(70°) = \cos(20°)$

$\tan(20°) = \cot(70°)$ and $\tan(70°) = \cot(20°)$

$\sec(20°) = \csc(70°)$ and $\sec(70°) = \csc(20°)$

THE ACUTE ANGLES IN A RIGHT TRIANGLE

A right triangle has two acute angles that are complementary to each other, and the co-function relationships can be seen using any right triangle:



$\sin(A) = \frac{4}{5} = \cos(B)$ and $\sin(B) = \frac{3}{5} = \cos(A)$

$\tan(A) = \frac{4}{3} = \cot(B)$ and $\tan(B) = \frac{3}{4} = \cot(A)$

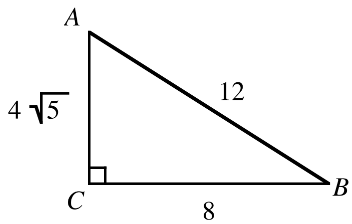
$\sec(A) = \frac{5}{3} = \csc(B)$ and $\sec(B) = \frac{5}{4} = \csc(A)$

These co-function relationships are useful when finding the six trigonometric function values for the two acute angles in a triangle. For example, once you know that $\tan(A) = \frac{\sqrt{5}}{3}$, then you also know that, without any work, $\cot(B) = \frac{\sqrt{5}}{3}$.

Section 2.2 Focus Exercises

For each triangle, find all six trig function values for each acute angle. Be sure to simplify each completely, including rationalizing any denominators that need it.

1.



$\sin(A)$ =

$\cos(A)$ =

$\tan(A)$ =

$\cot(A)$ =

$\sec(A)$ =

$\csc(A)$ =

$\sin(B)$ =

$\cos(B)$ =

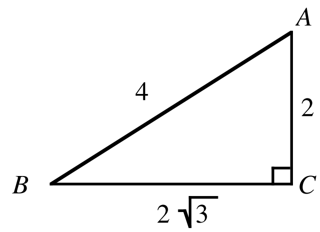
$\tan(B)$ =

$\cot(B)$ =

$\sec(B)$ =

$\csc(B)$ =

2.



$\sin(A)$ =

$\cos(A)$ =

$\tan(A)$ =

$\cot(A)$ =

$\sec(A)$ =

$\csc(A)$ =

$\sin(B)$ =

$\cos(B)$ =

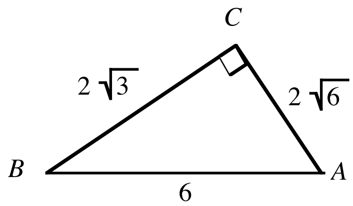
$\tan(B)$ =

$\cot(B)$ =

$\sec(B)$ =

$\csc(B)$ =

3.



$\sin(A) =$

$\cos(A) =$

$\tan(A) =$

$\cot(A) =$

$\sec(A) =$

$\csc(A) =$

$\sin(B) =$

$\cos(B) =$

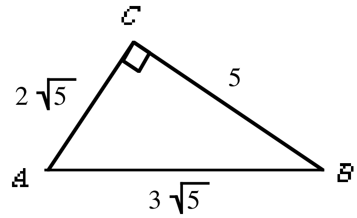
$\tan(B) =$

$\cot(B) =$

$\sec(B) =$

$\csc(B) =$

4.



$\sin(A) =$

$\cos(A) =$

$\tan(A) =$

$\cot(A) =$

$\sec(A) =$

$\csc(A) =$

$\sin(B) =$

$\cos(B) =$

$\tan(B) =$

$\cot(B) =$

$\sec(B) =$

$\csc(B) =$