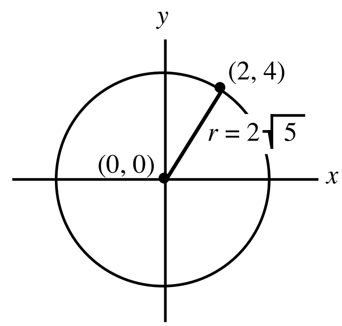
Section 2.5 Trigonometry in the *x*-*y*-Plane

Every point in the *x*-*y*-plane is on a circle



What circle is the point (2, 4) on? We start by finding the radius of the circle; use $x^2 + y^2 = r^2$ with (2, 4):

$$(2)^2 + (4)^2 = r^2$$

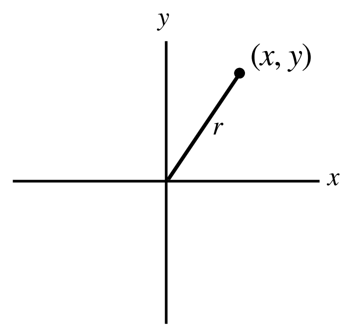
$$4 + 16 = r^2$$

$$20 = r^2$$

$$\pm\sqrt{20} = r$$

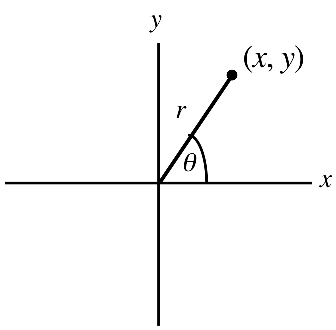
$$r = \sqrt{20} = 2\sqrt{5}$$

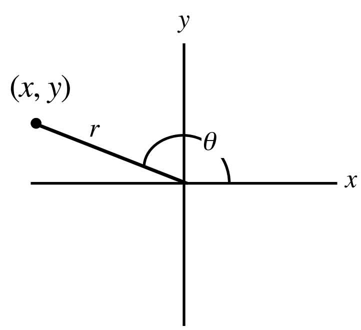
So, the point (2, 4) is on the circle $x^2 + y^2 = 20$

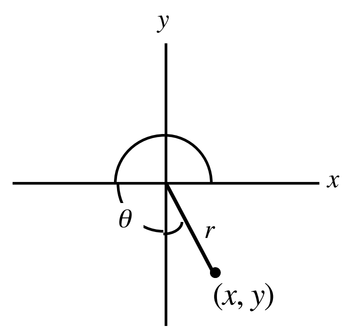


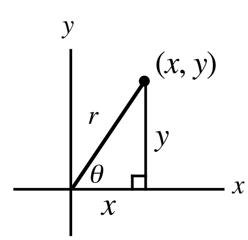
Without drawing a circle each time, we can identify any point in the *x*-*y*-plane as being connected to the origin by a radius. In general, this might look like the diagram at right.

This arbitrary point, (*x, y*) can be in any quadrant. Regardless of the quadrant, the angle formed by the given radius and the positive *x*-axis is called θ, the Greek letter *theta.*







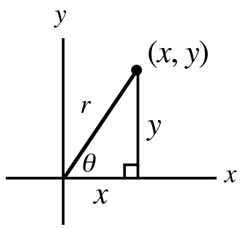


Considering a first quadrant point, (*x, y*), we can form a right triangle by drawing a line segment from the point perpendicular to the *x*-axis.

The hypotenuse of this right triangle is *r*, and the leg measures are the actual *x*- and *y*-values (coordinates) of the point.

Using this right triangle in the *x*-*y*-plane, allows us to define the trigonometric functions in a different way.

The *x*-*y*-plane Definitions of Sine, Cosine, and Tangent.



Using the right triangle (at left), we have the following definitions

$\sin(\theta) = \frac{y}{r}$ $\quad\quad$ $\cos(\theta) = \frac{x}{r}$ $\quad\quad$ $\tan(\theta) = \frac{y}{x}$

Note: r is found using $r^2 = x^2 + y^2$; also, r is always positive.

Example 1: Find the sine, cosine, and tangent of θ if the point (2, 4) is on the terminal side of θ.

Procedure: We have the x- and y-values, but we must have the value of r before we can find the sine and cosine values. We saw, at the beginning of this section, the point (2, 4) is on a circle with radius $r = 2\sqrt{5}$.

Answer: $x = 2$, $y = 4$, $r = 2\sqrt{5}$

Place these values into the new trig function definitions.

- $\sin(\theta) = \frac{y}{r} = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}} = \mathbf{\frac{2\sqrt{5}}{5}}$
- $\cos(\theta) = \frac{x}{r} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}} = \mathbf{\frac{\sqrt{5}}{5}}$
- $\tan(\theta) = \frac{y}{x} = \frac{4}{2} = \mathbf{2}$

These new x-y-plane definitions of the trigonometric functions allow us to consider sine, cosine, and tangent values for points in other quadrants, as demonstrated in this next example.

Example 2: Find the sine, cosine, and tangent of θ if the point (-3, 1) is on the terminal side of θ.

Answer:

$r^2 = x^2 + y^2$

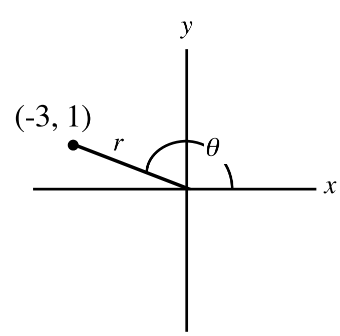
$r^2 = (-3)^2 + (1)^2$

$r^2 = 9 + 1 = 10$

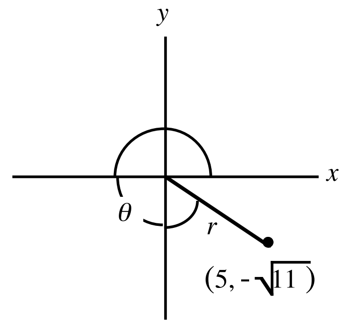
$r = \sqrt{10}$ ↗

→ $x = -3$, $y = 1$, $r = \sqrt{10}$,

- $\sin(\theta) = \frac{y}{r} = \frac{1}{\sqrt{10}} = \mathbf{\frac{\sqrt{10}}{10}}$
- $\cos(\theta) = \frac{x}{r} = \frac{-3}{\sqrt{10}} = \mathbf{\frac{-3\sqrt{10}}{10}}$
- $\tan(\theta) = \frac{y}{x} = \frac{1}{-3} = \mathbf{-\frac{1}{3}}$ or $\mathbf{\frac{-1}{3}}$



You Try It 1 Find the sine, cosine, and tangent of θ if the point $(5, -\sqrt{11})$ is on the terminal side of θ.



$r^2 = x^2 + y^2$

a) $\sin(\theta)$ =

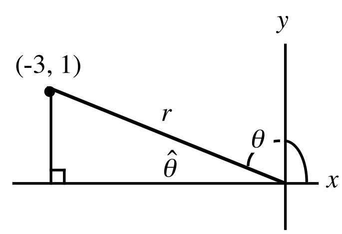
b) $\cos(\theta)$ =

c) $\tan(\theta)$ =

RIGHT TRIANGLES IN OTHER QUADRANTS

It appears as though we cannot form a right triangle in Quadrant II because θ is an obtuse angle. We can, though, form a right triangle by dropping a perpendicular line segment to the (negative) x-axis. The angle with the x-axis won't be θ. Instead, it is the supplement of θ.

We have a symbol for this acute angle, $\hat{\theta}$, which is called "theta-hat." $\hat{\theta}$ is always an acute angle and always has the x-axis as one of its sides. The other side is the radius that extends from the origin to the given point.

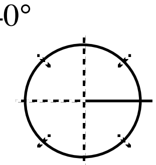


$\hat{\theta}$ is found only when making a right triangle with the x-axis, never to the y-axis, even in Quadrants III and IV.

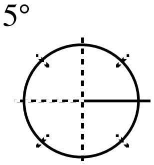
Note: The reason we use only the x-axis for creating right triangles in the x-y-plane is, in standard position, the initial side is always the positive x-axis.

You Try It 2 For each, locate θ on the circle and identify $\hat{\theta}$. *(Hint: Determine how many degrees the angle is from the nearest x-axis.)*

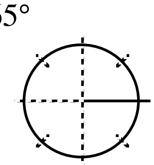
a) $\theta = 140°$



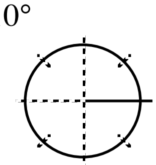
b) $\theta = 215°$



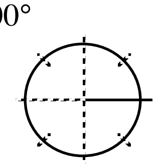
c) $\theta = 265°$



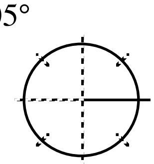
d) $\theta = 310°$



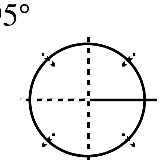
e) $\theta = 400°$



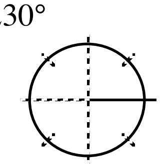
f) $\theta = 505°$



g) $\theta = -95°$



h) $\theta = -230°$

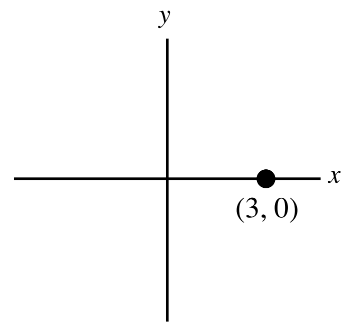


TRIGONOMETRIC VALUES FOR AXIAL POINTS

With these new definitions for sine, cosine, and tangent, we can now consider finding values such as sin(90°), cos(180°), and tan(0°).

Example 3: Find the following:

a) sin(0°) b) cos(0°) c) tan(0°)



Procedure: We can use any point on the positive *x*-axis. For this example, let's choose (3, 0). We must also identify the value of *r*: $r = 3$.

Answer:

a) $\sin(0°) = \frac{y}{r} = \frac{0}{3} = 0$ b) $\cos(0°) = \frac{x}{r} = \frac{3}{3} = 1$ c) $\tan(0°) = \frac{y}{x} = \frac{0}{3} = 0$

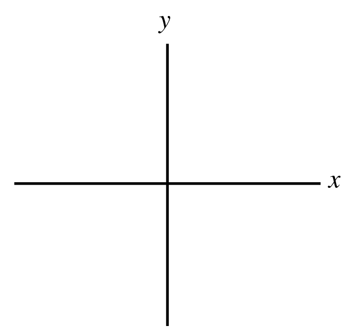
With these new sine, cosine, and tangent values (0 and 1), it is possible to get 0 (zero) in the denominator of a reciprocal function. If that happens, the function value is *Undefined.*

You Try It 3 Use the results in Example 3 to find the following:

a) csc(0°) b) sec(0°) c) cot(0°)

You Try It 4 Find the following:

(**Hint:** Use any point on the negative *y*-axis, and identify the value of *r*. Use Example 3 as a guide.)



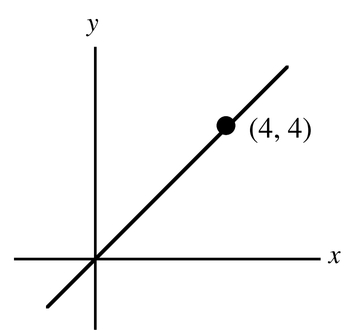
a) sin(270°) b) cos(270°)

c) tan(270°) d) cot(270°)

e) sec(270°) f) csc(270°)

You Try It 5 Using the point (4, 4), find the value of *r* and each of the following:

(**Note:** In general, we can use any Quad I point on the line $y = x$.)



a) sin(45°) b) cos(45°)

c) tan(45°) d) cot(45°)

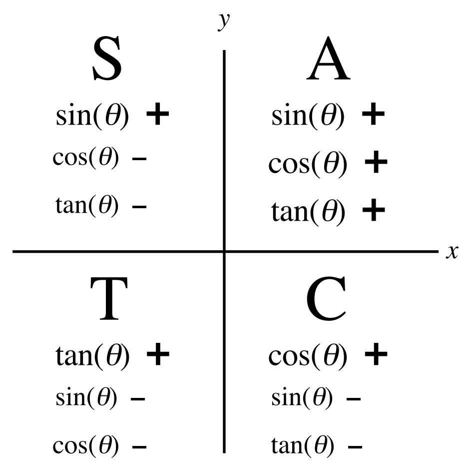
e) sec(45°) f) csc(45°)

POSITIVITY (AND NEGATIVITY) FOR NON-AXIAL POINTS (ASTC)

When θ terminates in a quadrant (not on an axis), the x- and y-values are either positive or negative, depending on the quadrant. The same is true for the corresponding sine and cosine values for that θ.

- In Quadrants I and II, the y-value is positive, and so is $\sin\theta$.
- In Quadrants III and IV, the y-value is negative, and so is $\sin\theta$.

A helpful way to remember the positivity (and negativity) of the three main trigonometric functions is to use the abbreviation **A-S-T-C**, as shown in the diagram and described below:



A — **ALL** functions are positive: Sine, Cosine, Tangent, and their reciprocals.

S — The **SINE** and **COSECANT** functions are positive

T — The **TANGENT** and **COTANGENT** functions are positive

C — The **COSINE** and **SECANT** functions are positive

A common acronym for this abbreviation is "All-Students-Take-Calculus" (even though it isn't quite true).

Example 4: Based on the given information, in which quadrant does θ terminate.

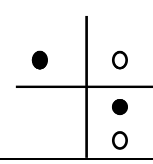
Note: Greater than zero (> 0) means "positive," and less than zero (< 0) means "negative."

a) $\tan\theta < 0$ and $\cos\theta > 0$ b) $\sin\theta < 0$ and $\cot\theta > 0$ c) $\sec\theta < 0$ and $\csc\theta > 0$

Procedure: For each function, place a dot in the two quadrants indicated by the inequality. The single quadrant that has two dots is the answer.

a) $\tan\theta < 0$: Tangent is negative in QII and **QIV**

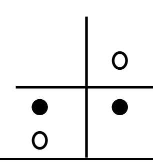
$\cos\theta > 0$: Cosine is positive in QI and **QIV**



Answer: **QIV**

b) $\sin\theta < 0$: Sine is negative in **QIII** and QIV

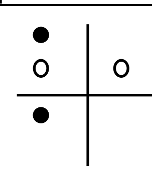
$\cot\theta > 0$: Cotangent is positive in QI and **QIII**



QIII

c) $\sec\theta < 0$: Secant is negative in **QII** and QIII

$\csc\theta > 0$: Cosecant is positive in QI and **QII**



QII

You Try It 6 Based on the given information, in which quadrant does θ terminate.

a) $\tan\theta > 0$ and $\sec\theta < 0$

b) $\sin\theta > 0$ and $\cot\theta < 0$

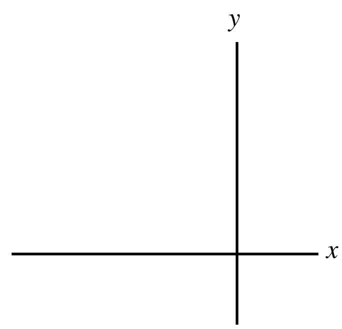
c) $\cos\theta > 0$ and $\csc\theta < 0$

d) $\sec\theta > 0$ and $\csc\theta > 0$

When θ terminates in a quadrant other than Quadrant I, such as 135°, we use ASTC to help us determine the positivity or negativity of the function value.

You Try It 7 Find the following:

(**Hint:** Use any Quad II point on the line $y = -x$, and identify the value of r. Refer to You Try It 4 as a guideline.)



a) The value of $\hat{\theta}$.

b) sin(135°)

c) cos(135°)

d) tan(135°)

e) cot(135°)

f) sec(135°)

g) csc(135°)

Using the *x, y,* and *r* definitions of sine, cosine, and tangent, we can find other trigonometric function values for an unspecified angle measure, θ.

Example 5: Given: θ terminates in QII and $\sec\theta = \frac{-\sqrt{13}}{2}$. Find $\sin\theta$, $\cos\theta$, and $\tan\theta$. Simplify.

Procedure: We must first identify the values of x, y, and r, We are given $\sec\theta = \frac{-\sqrt{13}}{2}$, and we can identify its reciprocal, $\cos\theta = \frac{2}{-\sqrt{13}} = \frac{x}{r}$. We nearly have the values of x and r, but because r must be positive, we should place the negative in the numerator:

$\cos\theta = \frac{-2}{\sqrt{13}} = \frac{x}{r}$ So, $x = -2$ and $r = \sqrt{13}$. Use these values to find y.

Note: Because θ terminates in QII, y must be positive.

Answer:

$r^2 = x^2 + y^2$

$(\sqrt{13})^2 = (-2)^2 + y^2$

$13 = 4 + y^2$

$9 = y^2$

$\sqrt{9} = \sqrt{y^2}$

$\pm 3 = y$ θ terminates in QII; y is positive.

So, $y = +3$ ↗

→ $x = -2$, $y = 3$, $r = \sqrt{13}$,

➢ $\sin\theta = \frac{y}{r} = \frac{3}{\sqrt{13}} = \mathbf{\frac{3\sqrt{13}}{13}}$

➢ $\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{13}} = \mathbf{\frac{-2\sqrt{13}}{13}}$

➢ $\tan\theta = \frac{y}{x} = \frac{3}{-2} = \mathbf{\frac{-3}{2}}$

You Try It 8 Given θ terminates in QIV and $\cot\theta = \frac{-\sqrt{7}}{3}$. Find the following.

Show work here:

a) $\sin\theta$

b) $\cos\theta$

c) $\tan\theta$

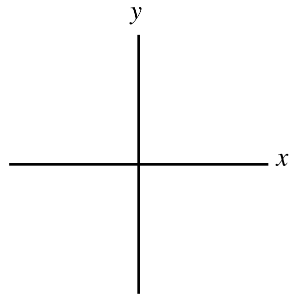
You Try It Answers

YTI 1:	a)	$-\frac{\sqrt{11}}{6}$	b)	$\frac{5}{6}$	c)	$-\frac{\sqrt{11}}{5}$		
YTI 2:	a)	$\hat{\theta} = 40°$	b)	$\hat{\theta} = 35°$	c)	$\hat{\theta} = 85°$	d)	$\hat{\theta} = 50°$
	e)	$\hat{\theta} = 40°$	f)	$\hat{\theta} = 35°$	g)	$\hat{\theta} = 85°$	h)	$\hat{\theta} = 50°$
YTI 3:	a)	undefined	b)	1	c)	undefined		
YTI 4:	a)	-1	b)	0	c)	undefined		
	d)	0	e)	undefined	f)	-1		
YTI 5:	a)	$\frac{\sqrt{2}}{2}$	b)	$\frac{\sqrt{2}}{2}$	c)	1		
	d)	1	e)	$\sqrt{2}$	f)	$\sqrt{2}$		
YTI 6:	a)	Quad III	b)	Quad II	c)	Quad IV	d)	Quad I
YTI 7:	a)	$\hat{\theta} = 45°$	b)	$\frac{\sqrt{2}}{2}$	c)	$-\frac{\sqrt{2}}{2}$	d)	-1
	e)	-1	f)	$-\sqrt{2}$	g)	$\sqrt{2}$		
YTI 8:	a)	$\frac{-3}{4}$	b)	$\frac{\sqrt{7}}{4}$	c)	$\frac{-3\sqrt{7}}{7}$		

Section 2.5 Focus Exercises

For each, plot the point in the x-y-plane and draw an angle in standard position, terminating at that point; label the angle as θ. Then find all six trigonometric functions of θ. Simplify.

1. (2, 6)



$\sin\theta =$

$\cos\theta =$

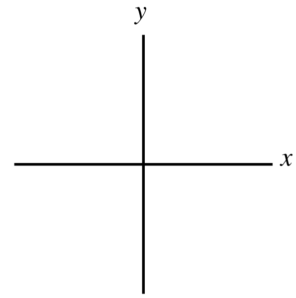
$\tan\theta =$

$\cot\theta =$

$\sec\theta =$

$\csc\theta =$

2. (4, -3)



$\sin\theta =$

$\cos\theta =$

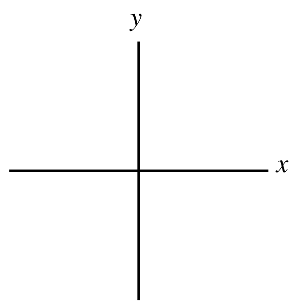
$\tan\theta =$

$\cot\theta =$

$\sec\theta =$

$\csc\theta =$

3. (0, -5)



$\sin\theta =$

$\cos\theta =$

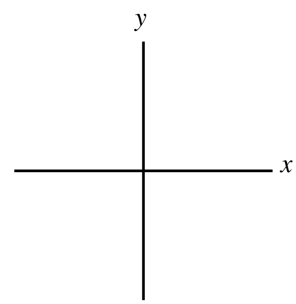
$\tan\theta =$

$\cot\theta =$

$\sec\theta =$

$\csc\theta =$

4. (3, 0)



$\sin\theta =$

$\cos\theta =$

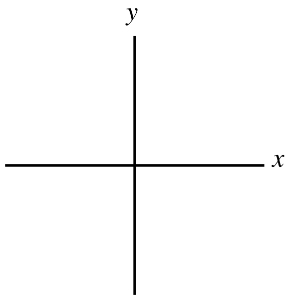
$\tan\theta =$

$\cot\theta =$

$\sec\theta =$

$\csc\theta =$

5. $\left(-\sqrt{7}, -3\right)$



$\sin\theta =$

$\cos\theta =$

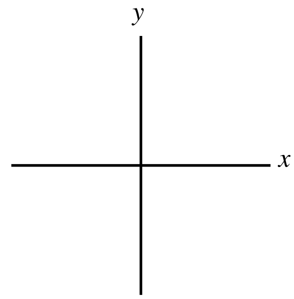
$\tan\theta =$

$\cot\theta =$

$\sec\theta =$

$\csc\theta =$

6. $\left(\sqrt{2}, -\sqrt{2}\right)$



$\sin\theta =$

$\cos\theta =$

$\tan\theta =$

$\cot\theta =$

$\sec\theta =$

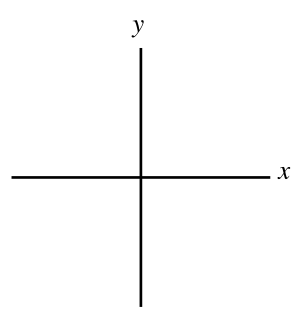
$\csc\theta =$

For each, identify $\hat{\theta}$.

7. $\theta = 110°$ **8.** $\theta = 280°$ **9.** $\theta = 155°$ **10.** $\theta = 275°$

11. $\theta = 380°$ **12.** $\theta = 615°$ **13.** $\theta = -145°$ **14.** $\theta = -350°$

Based on the given information, in which quadrant does θ terminate.
You may use the *x*-*y*-axes at right to assist you.



15. $\tan\theta < 0$ and $\cos\theta < 0$ **16.** $\sin\theta > 0$ and $\sec\theta > 0$

17. $\csc\theta > 0$ and $\tan\theta < 0$ **18.** $\cot\theta < 0$ and $\cos\theta > 0$

19. $\sin\theta > 0$ and $\cot\theta > 0$ **20.** $\csc\theta < 0$ and $\cos\theta > 0$

21. $\sec\theta < 0$ and $\cot\theta < 0$ **22.** $\tan\theta > 0$ and $\sin\theta < 0$

For each, based on the given information, find sin(θ), cos(θ), and tan(θ). Simplify.

23. $\sec\theta = -\frac{5}{4}$ and θ terminates in QII.

24. $\cot\theta = \frac{3}{4}$ and θ terminates in QI.

25. $\csc\theta = -\frac{6}{\sqrt{11}}$ and θ terminates in QIII.

26. $\sec\theta = \frac{2\sqrt{3}}{3}$ and θ terminates in QIV.

27. $\cot\theta = \frac{1}{3}$ and θ terminates in QIII.

28. $\csc\theta = \sqrt{5}$ and θ terminates in QII.

29. θ terminates in QIV and $\sec\theta = 4$

30. θ terminates in QI and $\cot\theta = 2\sqrt{2}$