Section 2.6 Trigonometric Identities

In this section we explore the various ways in which the trigonometric functions are related to each other.

Reciprocal Identities:

Recall from Section 2.2, these reciprocal identities:

- $\sec\theta = \dfrac{1}{\cos\theta}$ and
- $\cos\theta = \dfrac{1}{\sec\theta}$
- $\csc\theta = \dfrac{1}{\sin\theta}$ and
- $\sin\theta = \dfrac{1}{\csc\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta}$ and
- $\tan\theta = \dfrac{1}{\cot\theta}$

You Try It 1

a) If $\cos\theta = \frac{2}{7}$, then $\sec\theta =$

b) If $\sin\theta = \frac{1}{5}$, then $\csc\theta =$

c) If $\tan\theta = \frac{-2}{\sqrt{5}}$, then $\cot\theta =$

d) If $\csc\theta = \frac{\sqrt{7}}{2}$, then $\sin\theta =$

e) If $\cot\theta = \frac{1}{2}$, then $\tan\theta =$

f) If $\sec\theta = -\sqrt{2}$, then $\cos\theta =$

Two trigonometric functions, tangent and cotangent, are actually ratios of sine and cosine.

Ratio Identities: $\sin\theta$ compared to $\cos\theta$:

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Why does $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$?

Consider that $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$, then

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r}\cdot\frac{r}{x} = \frac{y}{x} = \tan\theta$$

You Try It 2

Given $\cos\theta = \frac{3}{4}$ and $\sin\theta = \frac{\sqrt{7}}{4}$, find

a) $\tan\theta$

b) $\cot\theta$

The Square of a Trigonometric Function:

The square of a trigonometric function, such as $(\sin\theta)^2$, is both a common and valuable expression.

We abbreviate this as $\sin^2\theta$. It is important to note that we are squaring the function, not the angle measure. When evaluating a squared trigonometric function, we must find the value of the function before squaring it.

For example, if $\sin\theta = \frac{3}{5}$, then $\sin^2\theta = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$.

You Try It 3

a) If $\cos\theta = \frac{2}{3}$, then $\cos^2\theta =$

b) If $\tan\theta = \frac{5\sqrt{2}}{6}$, then $\tan^2\theta =$

Pythagorean Identities:

There are also several trigonometric identities that are based on the Pythagorean Theorem. To start, keep in mind these three relationships: (i) $\sin\theta = \frac{y}{r}$; (ii) $\cos\theta = \frac{x}{r}$; and (iii) $x^2 + y^2 = r^2$.

Sine/Cosine Here is the development of the Pythagorean Identity for sine and cosine:

$$x^2 + y^2 = r^2$$ Divide each side by r^2.

$$\frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2}$$ Split the left side into two fractions and simplify the right side.

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$ Write the fractions as a quantity squared.

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$ Use the $\sin\theta$ and $\cos\theta$ definitions.

1. $\boxed{\cos^2\theta + \sin^2\theta = 1}$ This is the most basic form of the Pythagorean Identity.

We can solve this basic equation for $\cos^2\theta$:

1a) $\boxed{\cos^2\theta = 1 - \sin^2\theta}$

You Try It 4

Solve the basic equation for $\sin^2\theta$:

1b) $\boxed{}$

Another variation of the Pythagorean Identity is found by taking the square root of each side of Equation 1a),

1c) $\boxed{\cos\theta = \pm\sqrt{1 - \sin^2\theta}}$

You Try It 5

Take the square root of each side of equation 1b).

1d) $\boxed{}$

Example 1: Use the identity $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$ to find $\sin\theta$ when $\cos\theta = \frac{-2}{3}$. and θ terminates in QIII. Simplify.

Answer: Because θ terminates in QIII, $\sin\theta$ will be negative.

$$\sin\theta = -\sqrt{1 - \left(\frac{-2}{3}\right)^2} = -\sqrt{1 - \frac{4}{9}} = -\sqrt{\frac{5}{9}} = \frac{-\sqrt{5}}{\sqrt{9}} = \frac{-\sqrt{5}}{3}$$

You Try It 6 Use the identity $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$ to find $\cos\theta$ when $\sin\theta = \frac{-2\sqrt{6}}{7}$. and θ terminates in QIV. Simplify.

More Pythagorean Identities

Tangent/Secant

We can create another Pythagorean identity by dividing each term in the original (Equation 1) by $\cos^2\theta$:

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

2. $\boxed{1 + \tan^2\theta = \sec^2\theta}$

You Try It 7 Solve this identity for $\tan^2\theta$:

2a) $\tan^2\theta =$

Cotangent/Cosecant

You Try It 8 This time we can create another Pythagorean identity by dividing each term in the original by $\sin^2\theta$:

$$\cos^2\theta + \sin^2\theta = 1$$

3. $\boxed{}$

You Try It 9 Solve this identity for $\cot^2\theta$:

3a) $\cot^2\theta =$

You Try It Answers

YTI 1: a) $\frac{7}{2}$ b) 5 c) $-\frac{\sqrt{5}}{2}$ d) $\frac{2\sqrt{7}}{7}$

e) 2 f) $-\frac{\sqrt{2}}{2}$

YTI 2: a) $\frac{\sqrt{7}}{3}$ b) $\frac{3\sqrt{7}}{7}$

YTI 3: a) $\frac{4}{9}$ b) $\frac{25}{18}$

YTI 4: $\sin^2\theta = 1 - \cos^2\theta$

YTI 5: $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$

YTI 6: $\frac{5}{7}$

YTI 7: $\tan^2\theta = \sec^2\theta - 1$

YTI 8: $\cot^2\theta + 1 = \csc^2\theta$

YTI 9: $\cot^2\theta = \csc^2\theta - 1$

Section 2.6 Focus Exercises

Identify the value of the indicated reciprocal function. Simplify

1. If $\cot\theta = 6$, then $\tan\theta =$

2. If $\cos\theta = \frac{1}{2}$, then $\sec\theta =$

3. If $\sin\theta = \frac{2}{\sqrt{7}}$, then $\csc\theta =$

4. If $\csc\theta = \frac{4}{\sqrt{2}}$, then $\sin\theta =$

5. If $\tan\theta = \frac{-\sqrt{3}}{3}$, then $\cot\theta =$

6. If $\sec\theta = \sqrt{5}$, then $\cos\theta =$

7. If $\csc\theta = \frac{4\sqrt{2}}{3}$, then $\sin\theta =$

8. If $\cot\theta = \frac{2\sqrt{6}}{3}$, then $\tan\theta =$

9. If $\cos\theta = 0$, then $\sec\theta =$

10. If $\sin\theta = -1$, then $\csc\theta =$

Evaluate each.

11. Find $\sin^2\theta$ when $\sin\theta = \frac{2}{5}$.

12. Find $\cos^2\theta$ when $\cos\theta = -\frac{\sqrt{3}}{5}$.

13. Find $\tan^2\theta$ when $\tan\theta = -\frac{\sqrt{6}}{3}$.

14. Find $\cot^2\theta$ when $\cot\theta = \frac{\sqrt{2}}{4}$.

15. Find $\sec^2\theta$ when $\sec\theta = \frac{5\sqrt{2}}{6}$.

16. Find $\csc^2\theta$ when $\csc\theta = -\frac{3\sqrt{2}}{2}$.

Given the values of $\sin\theta$ and $\cos\theta$, find $\tan\theta$ and $\cot\theta$. Simplify.

17. $\sin\theta = \frac{24}{25}$ and $\cos\theta = \frac{7}{25}$

$\tan\theta =$ $\cot\theta =$

18. $\sin\theta = \frac{2}{3}$ and $\cos\theta = \frac{\sqrt{5}}{3}$

$\tan\theta =$ $\cot\theta =$

19. $\sin\theta = -\frac{\sqrt{15}}{4}$ and $\cos\theta = -\frac{1}{4}$

$\tan\theta =$ $\cot\theta =$

20. $\sin\theta = \frac{\sqrt{2}}{2}$ and $\cos\theta = \frac{\sqrt{2}}{2}$

$\tan\theta =$ $\cot\theta =$

21. $\sin\theta = -1$ and $\cos\theta = 0$

$\tan\theta =$ $\cot\theta =$

22. $\sin\theta = 0$ and $\cos\theta = 1$

$\tan\theta =$ $\cot\theta =$

Use the identity $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$ to answer the following. Simplify.

23. Find $\sin\theta$ when $\cos\theta = \frac{-3}{5}$.
and θ terminates in QII.

24. Find $\sin\theta$ when $\cos\theta = -\frac{\sqrt{2}}{4}$.
and θ terminates in QIII.

25. Find $\sin\theta$ when $\cos\theta = \frac{\sqrt{3}}{2}$.
and θ terminates in QI.

26. Find $\sin\theta$ when $\cos\theta = \frac{2}{3}$.
and θ terminates in QIV.