# Section 2.7 Proving Trigonometric Identities

In this section, we use the identities presented in Section 2.6 to do two different tasks:

1) to simplify a trigonometric expression, and

2) prove that a supposed identity really is an identity.

## SIMPLIFYING A TRIGONOMETRIC EXPRESSION

**Example 1:** Write each expression in terms of $\sin\theta$ and/or $\cos\theta$, and simplify.

a) $\sec\theta \cot\theta$ b) $\tan\theta + \sec\theta\sin\theta$

**Procedure:** Use the reciprocal and ratio identities to first write the expression in terms of only $\sin\theta$ and $\cos\theta$, and then simplify, if possible.

**Answer:** a) $\sec\theta \cot\theta$ Write each in terms of sine and cosine.

$= \dfrac{1}{\cos\theta} \cdot \dfrac{\cos\theta}{\sin\theta}$ Divide out $\cos\theta$.

$= \dfrac{1}{\sin\theta}$ This is the same as $\csc\theta$.

$= \csc\theta$ This expression cannot be simplified further.

b) $\tan\theta + \sec\theta \sin\theta$ Write each in terms of sine and cosine.

$= \dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\cos\theta} \cdot \sin\theta$ Multiply $\sin\theta$ to the second fraction.

$= \dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}$ These fractions have the same denominator; we can add them directly.

$= \dfrac{2\sin\theta}{\cos\theta}$ $\dfrac{\sin\theta}{\cos\theta}$ can simplify to $\tan\theta$.

$= 2\tan\theta$ This expression cannot be simplified further.

**You Try It 1** Write each expression in terms of $\sin\theta$ and/or $\cos\theta$, and simplify.

a) $\sec\theta \tan\theta \sin\theta$

b) $\csc\theta - \cot\theta$

In Example 1b) we added two fractions with the same denominator. If the fractions have different denominators, then we must first identify the least common denominator (LCD) and build up each fraction appropriately.

**Example 2:** Build up each fraction to have common denominators and simplify.

**Procedure:** Identify the LCD and multiply by a form of 1 to create the LCD in each fraction.

**Answer:** $\dfrac{1}{\sin\theta} - \dfrac{1}{\cos\theta}$ . The LCD is $\sin\theta\cos\theta$. Multiply each by a form of 1 to build up each fraction

$= \dfrac{1}{\sin\theta} \cdot \dfrac{\cos\theta}{\cos\theta} - \dfrac{1}{\cos\theta} \cdot \dfrac{\sin\theta}{\sin\theta}$ Simplify the respective fractions.

$= \dfrac{\cos\theta}{\sin\theta\cos\theta} - \dfrac{\sin\theta}{\sin\theta\cos\theta}$ Combine the fractions into one fraction.

$= \dfrac{\cos\theta - \sin\theta}{\sin\theta\cos\theta}$ This expression cannot be simplified further.

**Example 3:** In each expression build up the fractions to have a common denominator. Simplify.

a) $\dfrac{1}{\cos\theta} - \cos\theta$ b) $\dfrac{1}{\sin^2\theta} + \dfrac{1}{\cos^2\theta}$.

**Procedure:** Identify the LCD and build up each fraction to have that denominator.

**Answer:**

a) $\dfrac{1}{\cos\theta} - \cos\theta$

The LCD is $\cos\theta$. Write a 1 in the denominator of the second fraction and multiply by a form of 1 to build it up.

$= \dfrac{1}{\cos\theta} - \dfrac{\cos\theta}{1} \cdot \dfrac{\cos\theta}{\cos\theta}$

Multiply.

$= \dfrac{1}{\cos\theta} - \dfrac{\cos^2\theta}{\cos\theta}$

Combine the fractions into one.

$= \dfrac{1 - \cos^2\theta}{\cos\theta}$

The numerator is $\sin^2\theta$.

$= \dfrac{\sin^2\theta}{\cos\theta}$

This expression is one simplified form. Other simplified forms are possible.

b) $\dfrac{1}{\sin^2\theta} + \dfrac{1}{\cos^2\theta}$.

The LCD is $\sin^2\theta\cos^2\theta$. Multiply each by a form of 1 to build up the fractions.

$= \dfrac{1}{\sin^2\theta} \cdot \dfrac{\cos^2\theta}{\cos^2\theta} + \dfrac{1}{\cos^2\theta} \cdot \dfrac{\sin^2\theta}{\sin^2\theta}$

Simplify the respective fractions.

$= \dfrac{\cos^2\theta}{\sin^2\theta\cos^2\theta} + \dfrac{\sin^2\theta}{\sin^2\theta\cos^2\theta}$

Combine the fractions into one fraction.

$= \dfrac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta}$

The numerator is a Pythagorean identity; replace the numerator with 1.

$= \dfrac{1}{\sin^2\theta\cos^2\theta}$

**You Try It 2** In each expression build up the fractions to have a common denominator. Simplify. Hint: In expression b), the second fraction has a denominator of 1.

a) $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}$

b) $\dfrac{\cos^2\theta}{\sin\theta} - \sin\theta$

**Example 4:** Multiply and simplify.

a) $\sin\theta(\cos\theta - 2)$

b) $(2\cos\theta - 3)(\cos\theta + 4)$

**Procedure:** For each, distribute. You may use FOIL for b).

**Answer:**

a) $\sin\theta(\cos\theta - 2)$

$= \sin\theta\cos\theta - 2\sin\theta$

b) $(2\cos\theta - 3)(\cos\theta + 4)$

$= 2\cos^2\theta + 8\cos\theta - 3\cos\theta - 12$

$= 2\cos^2\theta + 5\cos\theta - 12$

**You Try It 3** Multiply and simplify.

a) $\sin\theta(\csc\theta - \sin\theta)$

b) $(\cos\theta + 1)^2$

**PROVING IDENTITIES**

An identity is set up to look like an equation. However, when we are attempting to prove an identity (a *supposed* identity) is true, then we cannot assume that the left and right sides are equivalent, so we are not allowed to treat them like equations. In other words, we cannot use rules that apply to equations, such as adding the same term to both sides of the equal sign.

Instead, we must manipulate only one side of the supposed identity to make it identical to the other side. Once that task is complete, the identity has been proven.

**Note:** Once an identity is proven, then we can write alternative identities using the rules of equations. For example,

because we know that $\cos^2\theta + \sin^2\theta = 1$ ...

we can solve for $\sin^2\theta$: $\sin^2\theta = 1 - \cos^2\theta$

**Example 5:** For each, demonstrate that the equation is an identity by transforming the left side (only) to be equivalent to the right side.

a) $\csc\theta \tan\theta = \sec\theta$ b) $\dfrac{\tan\theta}{\sec\theta} = \sin\theta$

**Procedure:** Start by writing the left side in terms of sine and cosine only.

**Answer:** a) $\csc\theta \tan\theta = \sec\theta$ Write the left side in terms of sine and cosine.

$\dfrac{1}{\sin\theta} \cdot \dfrac{\sin\theta}{\cos\theta} =$ Divide out $\sin\theta$.

$\dfrac{1}{\cos\theta} =$ This is $\sec\theta$. Finish the proof.

$\sec\theta = \sec\theta$ *QED**

b) $\dfrac{\tan\theta}{\sec\theta} = \sin\theta$ Write the left side in terms of sine and cosine.

$\dfrac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} =$ Change division to multiplying by the reciprocal.

$\dfrac{\sin\theta}{\cos\theta} \cdot \dfrac{\cos\theta}{1} =$ Divide out $\cos\theta$, and finish the proof.

$\sin\theta = \sin\theta$ *QED**

***Note:** It is common—though not required—to write "QED" at the end of a proof. It stands for the Latin phrase *Quod Erat Demonstrandum*; loosely translated, it says, "that which was to be demonstrated."

Some proofs are more involved and require using a variety of identities, including various forms of the Pythagorean identity.

**Example 6:** Demonstrate that the equation is an identity by transforming the left side (only) to be equivalent to the right side.

$$\csc\theta \tan\theta - \cos\theta = \frac{\sin^2\theta}{\cos\theta}$$

**Procedure:** Start by writing the left side in terms of sine and cosine only.

**Answer:** a) $\csc\theta \tan\theta - \cos\theta = \frac{\sin^2\theta}{\cos\theta}$ Write the left side in terms of sine and cosine.

$\frac{1}{\sin\theta} \cdot \frac{\sin\theta}{\cos\theta} - \cos\theta =$ Simplify the first fractions and write the second term as a fraction with 1 in the denominator.

$\frac{1}{\cos\theta} - \frac{\cos\theta}{1} =$ Get the common denominator: LCD = $\cos\theta$.

$\frac{1}{\cos\theta} - \frac{\cos\theta}{1} \cdot \frac{\cos\theta}{\cos\theta} =$ Simplify the second fraction.

$\frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta} =$ We must combine the fractions before we can use any other identities.

$\frac{1 - \cos^2\theta}{\cos\theta} =$ We can now use a Pythagorean identity in the numerator and finish the proof.

$\frac{\sin^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}$ *QED*

## You Try It Answers

**YTI 1:** a) $\tan^2\theta$ b) $\frac{1 - \cos\theta}{\sin\theta}$

**YTI 2:** a) $\frac{1}{\sin\theta \cos\theta}$ b) $\frac{\cos^2\theta - \sin^2\theta}{\sin\theta}$ (This cannot simplify further at this time.)

**YTI 3:** a) $\cos^2\theta$ b) $\cos^2\theta + 2\cos\theta + 1$

## Section 2.7 Focus Exercises

First write each expression in terms of only $\sin\theta$ and/or $\cos\theta$. Then simplify.

**1.** $\cos\theta\tan\theta$

**2.** $\tan\theta\sin\theta$

**3.** $\csc\theta\cos\theta\sin\theta$

**4.** $\dfrac{\csc\theta}{\tan\theta}$

**5.** $\dfrac{\cot\theta}{\sec\theta}$

**6.** $\dfrac{\tan\theta}{\sin\theta}$

**7.** $\tan\theta + \sec\theta$

**8.** $\cot\theta + \csc\theta\cos\theta$

**9.** $\sin\theta\sec\theta - \tan\theta$

**10.** $\sec\theta - \tan\theta\sin\theta$

For each, identify the least common denominator (LCD) and use it to build up each fraction and combine the fractions. Write all answers in terms of $\sin\theta$ and $\cos\theta$. Simplify.

**11.** $\dfrac{1}{\cos\theta} + \dfrac{1}{\cos^2\theta}$.

**12.** $\dfrac{1}{\sin\theta} - \sin\theta$

**13.** $\dfrac{1}{\sin\theta\cos\theta} - \dfrac{\cos\theta}{\sin\theta}$.

**14.** $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}$.

Multiply and simplify

**15.** $\cos\theta\,(\tan\theta - \cos\theta)$

**16.** $\csc\theta\,(\tan\theta + \sin\theta)$

**17.** $(\cos\theta + 2)(\cos\theta + 1)$

**18.** $(\cos\theta + \sin\theta)^2$

For each, demonstrate that the equation is an identity by transforming the left side (only) to be equivalent to the right side.

**19.** $\sec\theta \sin\theta \cos\theta = \sin\theta$

**20.** $\sec\theta \cot\theta = \csc\theta$

**21.** $\dfrac{\cos\theta}{\cot\theta} = \sin\theta$

**22.** $\dfrac{\sec\theta}{\csc\theta} = \tan\theta$

**23.** $\dfrac{\sin^2\theta}{\tan\theta} = \sin\theta\cos\theta$

**24.** $\csc\theta - \cos\theta\cot\theta = \sin\theta$

**25.** $\dfrac{\sin\theta}{\csc\theta} + \dfrac{\cos\theta}{\sec\theta} = 1$

**26.** $\tan\theta\sin\theta = \sec\theta - \cos\theta$