A.2 Factoring Trinomials

FACTORING QUADRATICS

A *quadratic* polynomial is in the form $ax^2 + bx + c$. In this section we discuss a one-step method for factoring trinomials such as

a) $2x^2 + 7x + 6$ b) $3w^2 - 20w + 12$ c) $6m^2 + 7m - 5$

$= (2x + 3)(x + 2)$ $= (w - 6)(3w - 2)$ $= (3m + 5)(2m - 1)$

Because factoring is the reverse process of multiplication, we can think about reversing the steps of multiplying binomials and apply it to factoring trinomials.

For example, consider multiplying $(3x + 2)(x - 6)$. We start by finding the four **FOIL** products:

$(3x + 2)(x - 6)$

F O I L

The four **FOIL** products: $= 3x^2 \underline{- 18x + 2x} - 12$

The two middle terms are like terms and can combine, giving us a trinomial: $= 3x^2 \underline{- 16x} - 12$

The reverse of this includes the following steps. Starting with a trinomial $3x^2 - 16x - 12$, write the middle term as two like terms. This will create a four-term polynomial which we can factor using factoring by grouping:

Start with a trinomial: $3x^2 \underline{- 16x} - 12$

Split the middle term into two terms and write the trinomial with four terms: $= 3x^2 \underline{- 18x + 2x} - 12$

Use factoring by grouping: $= (3x^2 - 18x) + (2x - 12)$

$= 3x(x - 6) + 2(x - 6)$

and here is the factored form: $= (3x + 2)(x - 6)$

The most important question in all of this is, how do we know what two like terms (in this case $-18x$ and $+2x$) to write in place of the middle term, $-16x$? The answer to this question is, the Factor Game.

THE FACTOR GAME

The Factor Game is a game of two numbers: a **Product** number and a **Sum** number. The game is to find a factor pair of the Product number that add to get the Sum number. The factor pair that works is called the "winning combination," or just "combination."

Example 1: Find the winning combination of the Factor Game with the given product and sum numbers.

a) Product = +20 and sum = -9. b) Product = -30 and sum = +1

Procedure: For a), the product is positive and the sum is negative so both factors in the factor pair must be negative. For b), the product is negative, so one factor must be positive and the other factor must be negative.

Answer: a) Factor pairs of +20: b) Factor pairs of -30:

+20		Sum	
-1	-20	-21	
-2	-10	-12	
-4	-5	-9	← This is it!

-30		Sum	
-1	+30	+29	
-2	+15	+13	
-3	+10	+7	
-5	+6	+1	← This is it!

You Try It 1 Find the winning combination of the Factor Game with the given product and sum numbers. Use Example 1 as a guide.

a) product = +32
sum = -12

b) product = -36
sum = +16

c) product = -40
sum = -6

TRINOMIALS AND THE FACTOR GAME

Now let's put the Factor Game to good use. Consider a trinomial of the form $ax^2 + bx + c$, where a, b, and c are integers, and $a > 0$.

In the trinomial $ax^2 + bx + c$, the Product number is $a \cdot c$, and the Sum number is b.

Example 2: Given the trinomial, identify the Product and Sum numbers, and find the winning combination of the Factor Game, if there is one.

a) $3x^2 - 14x + 8$ b) $5y^2 - 4y - 12$ c) $3w^2 + 6w - 8$

Procedure: Identify the values of a, b, and c and find the product and sum numbers of the Factor Game. Note: If a trinomial has no winning combination, then it is not factorable.

Answer:

a) Product = 3(8) = 24
Sum = -14

Combination : **-12 and -2**

b) Product = 5(-12) = -60
Sum = -4

Combination : **6 and -10**

c) Product = 3(- 8) = - 24
Sum = 6

There is **no combination.**

You Try It 2 Given the trinomial, identify the Product and Sum numbers, and find the winning combination for the Factor Game, if there is one. Use Example 2 as a guide.

a) $4x^2 + 12x + 5$ Product =

Sum =

Combination: ____________

b) $5x^2 - 13x + 6$ Product =

Sum =

Combination: ____________

c) $6x^2 - 13x - 4$ Product =

Sum =

Combination: ____________

d) $3x^2 + 4x - 4$ Product =

Sum =

Combination: ____________

FACTORING TRINOMIALS USING THE FACTOR GAME

The whole point of the Factor Game is to be able to rewrite a trinomial into four terms. It is the winning combination of the Factor Game that tells us how to split up the trinomial's middle term into two terms, thereby making it a four-term polynomial.

Consider factoring the trinomial $8x^2 + 10x - 3$. Here are the steps to factoring a trinomial using the Factor Game and factoring by grouping.

(1) Identify the **Product** and **Sum** numbers and find the combination to the Factor Game:

$8x^2 + 10x - 3$ Product $= 8(-3) = -24$, Sum $= +10$ The combination is +12 and -2.

The two factors in the combination are the coefficients of the two new x-terms that make the trinomial into a four-term polynomial. In other words, the middle term, $+ 10x$, can be replaced by $+ 12x - 2x$:

$$8x^2 \underline{+ 10x} - 3$$

(2) Write the trinomial with four terms: $= 8x^2 \underline{+ 12x - 2x} - 3$

(3) Use factoring by grouping: $= (8x^2 + 12x) + (-2x - 3)$

$$= 4x\mathbf{(2x + 3)} + -1\mathbf{(2x + 3)}$$

$$= (4x - 1)\mathbf{(2x + 3)}$$

We can verify that this factoring is true by multiplying the binomials together using **FOIL**.

If the Factor Game has no winning combination, then the trinomial is not factorable, and we say that it is *prime*.

Example 3: Factor each trinomial by using the Factor Game to rewrite it as a four-term polynomial. Then use factoring by grouping.

a) $6p^2 + 5p - 6$ b) $5x^2 - 16x + 12$ c) $3r^2 + 6r - 8$

Procedure: Identify the Product and Sum numbers and play the Factor Game. Use the winning combination to write the trinomial as a four-term polynomial, and use factoring by grouping. Verify that the result is true by using **FOIL** to multiply the results.

Answer:

a) $6p^2 + 5p - 6$ Product = 6(-6) = -36
Sum = + 5 The combination is -4 and +9.

$6p^2 + 5p - 6$ Split the middle term into $-4p + 9p$:

$= 6p^2 - 4p + 9p - 6$ Now show the groupings. (There must be a plus sign between the groupings.

$= (6p^2 - 4p) + (9p - 6)$ Factor out the common monomial factor from each group …

$= 2p(3p - 2) + 3(3p - 2)$ … and factor out $(3p - 2)$.

$= \mathbf{(2p + 3)(3p - 2)}$ Now verify this factoring by multiplying the binomials.

b) $5x^2 - 16x + 12$ Product = 5(12) = 60
Sum = - 16 The combination is -10 and -6.

$5x^2 - 16x + 12$ Split the middle term into $-10x - 6x$:

$= 5x^2 - 10x - 6x + 12$ Now show the groupings. (There must be a plus sign between the groupings.

$= (5x^2 - 10x) + (-6x + 12)$ Factor out the common monomial factor from each group.

$= 5x(x - 2) + -6(x - 2)$ Now factor out $(x - 2)$:

$= \mathbf{(5x - 6)(x - 2)}$ Now verify this factoring by multiplying the binomials.

c) $3r^2 + 6r - 8$ Product = 3(-8) = -24
Sum = + 6 There is **no combination**.

Because there is no combination to the Factor Game, the trinomial is ***prime***.

You Try It 3 Factor each trinomial by using the Factor Game to rewrite it as a four-term polynomial. Then use factoring by grouping. If the trinomial cannot be factored, write "prime." Use Example 3 as a guide.

a) $3y^2 + 11y + 6$

b) $2x^2 - 5x + 4$

c) $m^2 + m - 30$

d) $6p^2 - p - 2$

ONE-STEP TRINOMIAL FACTORING

Let's take another look at the trinomial in Example 3a):

$6p^2 + 5p - 6$ The winning combination is <u>-4 and +9</u>, so the middle term is split into $-4p + 9p$

F O I L

$= 6p^2 - 4p + 9p - 6$ Notice that the **F** product, $\mathbf{6p^2}$, and the **O** product, $\mathbf{-4p}$, are paired up in $\mathbf{(6p^2 - 4p)}$.

$= \mathbf{(6p^2 - 4p)} + (9p - 6)$

$= \mathbf{2p}(3p - 2) + 3(3p - 2)$ Notice also that their greatest common factor (GCF), $\mathbf{2p}$, is the first term of the first binomial factor (in the answer).

$= (\mathbf{2p} + 3)(3p - 2)$

This bit of information will serve as the foundation for factoring a trinomial in one step. We must now learn how to find the other three terms of the binomial factors.

For example, factor $8x^2 + 10x - 3$ in just one step:

A. Identify the four FOIL products within the trinomial, and play the Factor Game.

We already have the F and L products: **F** **L**

$$\mathbf{8x^2} + 10x \mathbf{- 3}$$

It's just a matter of finding the **O** and **I** products from the Factor Game:

$8x^2 + 10x - 3$ Product $= 8(-3) = -24$, Sum $= +10$ The combination is +12 and -2.

Either $12x$ or $-2x$ can be the **O** product, but it is common to choose the larger one to be **O**. ➔ {Let's choose the **O** product = $\mathbf{+12x}$ and the lesser as the **I** product = $\mathbf{-2x}$}

The <u>GCF of the **F** and **O** products is</u> $\mathbf{4x}$, and that is the first term of the first binomial factor.

B. Create the framework; Fill in the binomial factors; Verify the factoring

1. Because the trinomial is factorable, it will factor into two binomials, and we can write two sets of parentheses in anticipation of those binomial factors:

$$8x^2 + 10x - 3$$
$$(\quad)(\quad)$$

2. Instead of writing the **O**uter and **I**nner products as

$8x^2$ <u>$+ 12x - 2x$</u> $- 3$, we write the **O** product above the parentheses, as shown.

$$8x^2 + 10x - 3$$
$$+12x$$
$$(\quad)(\quad)$$

3. The first term of the first binomial is the GCF of the **F** and **O** products. The GCF of $8x^2$ and $12x$ is $4x$.

Now that we have the "first of the first" we can use it and the **FOIL** products to find other terms within the two binomials.

$$8x^2 + 10x - 3$$
$$+12x$$
GCF
$$(4x \quad)(\quad)$$

4. We can use the first term, **4x**, and the **F** product, $\mathbf{8x^2}$, to help us find the first term of the *second* binomial.

F $8x^2 + 10x - 3$
$$(4x \quad)(2x \quad)$$

5. Next, we find the **O** product. We already have the first term of the **O** product, $4x$. To create the **O** product, $+12x$, the second term of the second binomial must be +3.

O $+12x$
$$(4x \quad)(2x + 3)$$

6. To find the only remaining unknown term, we can use the **L** product, -3. This is the product of the two constant terms. Because one of the constant terms is already known, +3, the other must be -1, and this is the value that is placed in the first binomial.

L $8x^2 + 10x - 3$
$$(4x - 1)(2x + 3)$$

7. To complete the factoring, we must verify that it is accurate by multiplying the two binomials together. This can be done mentally or on paper using **FOIL**.

$$(4x - 1)(2x + 3)$$
$$= 8x^2 + 12x - 2x - 3$$
$$= 8x^2 + 10x - 3$$

One final note about factoring trinomials:

If the two factors in the combination of the factor game are exactly the same, then the trinomial is a perfect square trinomial.

For example, for $25y^2 + 20y + 4$, we get Product $= +100$, Sum $= +20$. The combination is <u>+10 and +10</u>.

and $25y^2 + 20y + 4$ factors into $(5y + 2)(5y + 2)$ which can be written as $(5y + 2)^2$.

You Try It 4 Factor each using the one-step method. Use the discussion above as a guide.

a) $10x^2 - 11x + 3$

b) $8m^2 - 2m - 3$

c) $10r^2 + 7r - 6$

d) $x^2 - 18x + 81$

e) $9y^2 - 6y + 1$

f) $9y^2 + 9y - 4$

THE DIFFERENCE OF SQUARES

Just as a pair of conjugates multiplies to the difference of squares, we can factor the difference of squares into a pair of conjugates. For example,

Examples:

$x^2 - 25$
$= (x + 5)(x - 5)$

$4y^2 - 81w^2$
$= (2y - 9w)(2y + 9w)$

$w^4 - 16$
$= (w^2 + 4)(w^2 - 4)$
$= (w^2 + 4)(w + 2)(w - 2)$

Note: The sum of squares is not factorable; it is prime. For example, $x^2 + 4$ is prime.

You Try It 5 Factor each binomial. Use the discussion above as a guide.

a) $x^2 - 100$

b) $4m^2 - 49$

c) $25x^2 + 1$

d) $y^3 - 36y$

e) $3p^3 - 75p$

f) $x^4 - 81$

You Try It Answers

YTI 1: a) -4 and -8 b) +18 and -2 c) -10 and +4

YTI 2:
a) Product = +20, Sum = +12; combination is +10 and +2
b) Product = +30, Sum = -13; combination is -10 and -3
c) Product = -24, Sum = -13; There is no combination.
d) Product = -12, Sum = 4; combination is -2 and +6

YTI 3:
a) $(3y + 2)(y + 3)$ b) Prime.
c) $(m - 5)(m + 6)$ d) $(3p - 2)(2p + 1)$

YTI 4:
a) $(5x - 3)(2x - 1)$ b) $(4m - 3)(2m + 1)$
c) $(2r - 1)(5r + 6)$ d) $(x - 9)^2$
e) $(3y - 1)^2$ f) $(3y - 1)(3y + 4)$

YTI 5:
a) $(x + 10)(x - 10)$ b) $(2m - 7)(2m + 7)$
c) Prime d) $y(y - 6)(y + 6)$
e) $3p(p + 5)(p - 5)$ f) $(x^2 + 9)(x + 3)(x - 3)$

A.2 Focus Exercises

Factor each trinomial by using the Factor Game. You may use any method presented in this section. If the trinomial is not factorable, write "prime."

1. $4x^2 + 11x + 6$

2. $2w^2 - 9w + 9$

3. $10y^2 + 9y + 2$

4. $p^2 - 12p + 36$

5. $5v^2 - 6v - 8$

6. $4x^2 - 12x + 5$

7. $4m^2 + 6m + 9$

8. $12k^2 - 8k + 1$

9. $h^2 - 16h + 64$

10. $10x^2 + 3x - 4$

11. $15r^2 + 4r - 3$

12. $8p^2 + 13p - 6$

13. $10x^2 - 17x + 6$

14. $15y^2 + 4y - 4$

15. $20m^2 - 11m - 3$

16. $30w^2 - 23w + 2$

Factor each binomial. *(Hint: Some of these require more than one factoring.)*

17. $x^2 - 144$

18. $9m^2 - 64$

19. $49x^2 - 1$

20. $x^2 + 100$

21. $81x^2 - 16$

22. $16x^4 - 1$

23. $w^3 - 25w$

24. $5m^3 - 45m$