A.4 Rationalizing the Denominator

RATIONALIZING THE DENOMINATOR

If a radical expression contains an irrational denominator, such as $\frac{\sqrt{7}}{\sqrt{3}}$, $\sqrt{\frac{7}{12}}$, or $\frac{20}{\sqrt{5}}$, then it is not considered to be in a simplified form. To simplify such a fraction, we use a process called *rationalizing the denominator*. To rationalize a denominator means to make it into a rational number—specifically, a whole number.

To rationalize a denominator requires us to create a perfect square radicand in the denominator. We do this by multiplying the fractions by a creative form of 1, $\frac{\sqrt{c}}{\sqrt{c}}$.

For example, we can multiply $\frac{\sqrt{7}}{\sqrt{3}}$ by $\frac{\sqrt{3}}{\sqrt{3}}$ to create $\sqrt{9}$ in the denominator, and we simplify it from there: $\frac{\sqrt{7}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{21}}{\sqrt{9}} = \frac{\sqrt{21}}{3}$

In the example above, we say that the **common multiplier** is $\sqrt{3}$, but it means that we must multiply both the numerator and the denominator by $\sqrt{3}$.

Note: Radicals are compatible with multiplication and division, but radicals are incompatible with addition and subtraction:

[1] $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$ and [2] $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

Likewise, [3] $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ and [4] $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

However, $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$ and $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

Example 1: Rationalize the denominator and simplify each expression.

a) $\frac{\sqrt{3}}{\sqrt{6}}$ b) $\frac{20}{\sqrt{5}}$

Procedure: Multiply by an appropriate form of 1 to create a perfect square radicand in the denominator. Simplify if possible.

Answer: a) $\frac{\sqrt{3}}{\sqrt{6}} = \frac{\sqrt{3}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{18}}{\sqrt{36}} = \frac{\sqrt{9 \cdot 2}}{6} = \frac{\sqrt{9} \cdot \sqrt{2}}{6} = \frac{3\sqrt{2}}{6} = \frac{\sqrt{2}}{2}$

b) $\frac{20}{\sqrt{5}} = \frac{20}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{20\sqrt{5}}{\sqrt{25}} = \frac{20\sqrt{5}}{5} = 4\sqrt{5}$

Sometimes we can simplify the fraction before rationalizing. For example, in 1a), we could have made $\frac{\sqrt{3}}{\sqrt{6}}$ into a single radical: $\sqrt{\frac{3}{6}}$ and then simplify it to $\sqrt{\frac{1}{2}}$ and then $\frac{1}{\sqrt{2}}$ before rationalizing the denominator.

Sometimes the denominator can be simplified before rationalizing. For example, the $\sqrt{12}$ in the denominator of $\frac{\sqrt{5}}{\sqrt{12}}$ can first be simplified to $\sqrt{4 \cdot 3} = 2\sqrt{3}$:

$$\frac{\sqrt{5}}{\sqrt{12}} = \frac{\sqrt{5}}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{15}}{2\sqrt{9}} = \frac{\sqrt{15}}{2 \cdot 3} = \frac{\sqrt{15}}{6}$$

You Try It 1 Rationalize the denominator and simplify each expression. Use Example 1 as a guide.

a) $\frac{\sqrt{7}}{\sqrt{2}}$

b) $\frac{12}{\sqrt{3}}$

c) $\frac{\sqrt{3}}{\sqrt{8}}$

d) $\frac{12}{\sqrt{18}}$

WHY WE RATIONALIZE THE DENOMINATOR

We know *how* to rationalize the denominator, but *why* do we do it? To understand this, let's consider the decimal approximation of $\sqrt{2}$:

$$\sqrt{2} \approx 1.4142$$

In trigonometry the number $\frac{1}{\sqrt{2}}$ appears quite a bit. Where is this number located on the number line? Is it more than 1 or less than 1?

To answer these questions, we must find its decimal approximation. One way is to use long division:

$$\frac{1}{\sqrt{2}} \approx \frac{1}{1.4142} \rightarrow 1.4142\,\overline{)1.000000} \quad \text{(shown at right)}$$

The long division tells us $\frac{1}{\sqrt{2}} \approx 0.7071$

$$\begin{array}{r} 0.7071....\\ 1.4142\,\overline{)1.0000000} \\ -\ 98994 \\ \hline 100600 \\ -\ 98994 \\ \hline 16060 \\ -\ 14142 \\ \hline 1918 \end{array}$$

However, the process is much simpler if we first rationalize the denominator:

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx \frac{1.4142}{2}$$

This division is much simpler because the divisor is a whole number (shown at right):

This long division also tells us $\frac{\sqrt{2}}{2} \approx 0.7071$

$$\begin{array}{r} 0.7071....\\ 2\,\overline{)1.414200} \\ -14 \\ \hline 014 \\ -\ 14 \\ \hline 02 \\ -\ 02 \\ \hline 0 \end{array}$$

Example 2: Given $\sqrt{11} \approx 3.3166$, find the value of $\frac{1}{\sqrt{11}}$.

Answer: Rationalize the denominator and then use long division.

$$\frac{1}{\sqrt{11}} \cdot \frac{\sqrt{11}}{\sqrt{11}} = \frac{\sqrt{11}}{11}$$

So, $\frac{1}{\sqrt{11}} \approx 0.3015$

$$\begin{array}{r} 0.3015...\\ 11\,\overline{)3.3166} \\ -33 \\ \hline 016 \\ -\ \ 11 \\ \hline 56 \\ -\ 55 \\ \hline 1 \end{array}$$

RATIONALIZING WITH A BINOMIAL DENOMINATOR

The product of a binomial radical expression and its conjugate always results in a rational number, oftentimes an integer. For example,

1. The conjugate of $(\sqrt{7} + \sqrt{3})$ is $(\sqrt{7} - \sqrt{3})$ and their product is

$$(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3}) = (\sqrt{7})^2 - (\sqrt{3})^2 = 7 - 3 = 4$$

2. The conjugate of $(2 - \sqrt{10})$ is $(2 + \sqrt{10})$ and their product is

$$(2 + \sqrt{10})(2 - \sqrt{10}) = 2^2 - (\sqrt{10})^2 = 4 - 10 = -6$$

This knowledge is useful when a radical expression has a binomial in the denominator, such as $\dfrac{10}{3-\sqrt{7}}$. To rationalize the denominator, we use the conjugate of the denominator as the common multiplier. For example, to rationalize the denominator of $\dfrac{10}{3-\sqrt{7}}$, the common multiplier is $3+\sqrt{7}$, so we multiply the fraction by $\dfrac{3+\sqrt{7}}{3+\sqrt{7}}$:

$$\frac{10}{3-\sqrt{7}} \cdot \frac{3+\sqrt{7}}{3+\sqrt{7}} = \frac{10(3+\sqrt{7})}{9-7} = \frac{10(3+\sqrt{7})}{2} = 5(3+\sqrt{7}) = 15 + 5\sqrt{7}$$

Example 3: Rationalize the denominator and simplify $\dfrac{1-\sqrt{3}}{2+\sqrt{3}}$

Procedure: Multiply by an appropriate form of 1 to create a perfect square radicand in the denominator. The common multiplier is the conjugate of the denominator.

Answer:

$\dfrac{1-\sqrt{3}}{2+\sqrt{3}}$	Multiply by $\dfrac{2-\sqrt{3}}{2-\sqrt{3}}$.
$= \dfrac{1-\sqrt{3}}{2+\sqrt{3}} \cdot \dfrac{2-\sqrt{3}}{2-\sqrt{3}}$	Multiply the fractions. Multiplying the denominators creates the difference of squares.
$= \dfrac{(1-\sqrt{3}) \cdot (2-\sqrt{3})}{(2)^2 - (\sqrt{3})^2}$	Multiply out the numerator and simplify the denominator.
$= \dfrac{2-\sqrt{3}-2\sqrt{3}+\sqrt{9}}{4-3}$	Simplify the denominator as well as the numerator radicals.
$= \dfrac{2-3\sqrt{3}+3}{1}$	The numerator does not simplify any further.
$= 5 - 3\sqrt{3}$	Whew!

You Try It 2 Rationalize the denominator and simplify each expression. Use Example 3 as a guide.

a) $\dfrac{-8}{1 - \sqrt{5}}$

b) $\dfrac{\sqrt{2} - \sqrt{8}}{2 - \sqrt{3}}$

You Try It Answers

You Try It 1: a) $\dfrac{\sqrt{14}}{2}$ b) $4\sqrt{3}$ c) $\dfrac{\sqrt{6}}{4}$ d) $2\sqrt{2}$

You Try It 2: a) $2 + 2\sqrt{5}$ b) $-2\sqrt{2} - \sqrt{6}$

A.4 Focus Exercises

For each, identify the conjugate of the binomial and then find the product of the pair of conjugates.

1. $2 - \sqrt{3}$

2. $1 + \sqrt{5}$

3. $\sqrt{6} - \sqrt{2}$

4. $3\sqrt{2} + \sqrt{7}$

5. $\sqrt{10} - 2\sqrt{2}$

6. $2\sqrt{5} + 3\sqrt{5}$

For each, rationalize the denominator and simplify the expression.

7. $\dfrac{12}{\sqrt{3}}$

8. $\dfrac{\sqrt{8}}{\sqrt{5}}$

9. $\dfrac{\sqrt{15}}{\sqrt{6}}$

10. $\dfrac{\sqrt{5}}{\sqrt{8}}$

11. $\dfrac{\sqrt{7}}{\sqrt{18}}$

12. $\dfrac{20}{\sqrt{50}}$

Given the approximate value of the radical, find the value of the fraction by first rationalizing the denominator and then using long division. Round your result to three or four decimal places.

13. $\sqrt{3} \approx 1.732$; use this to find the approximate value of $\dfrac{1}{\sqrt{3}}$

14. $\sqrt{5} \approx 2.236$; use this to find the approximate value of $\dfrac{1}{\sqrt{5}}$.

15. $\sqrt{6} \approx 2.4495$; use this to find the approximate value of $\dfrac{1}{\sqrt{6}}$.

16. $\sqrt{10} \approx 3.1623$; use this to find the approximate value of $\dfrac{1}{\sqrt{10}}$.

For each, rationalize the denominator and simplify the expression.

17. $\dfrac{5}{4 + \sqrt{11}}$

18. $\dfrac{1}{1 + \sqrt{2}}$

19. $\dfrac{\sqrt{5}}{2 - \sqrt{5}}$

20. $\dfrac{\sqrt{2} - \sqrt{8}}{2 - \sqrt{3}}$