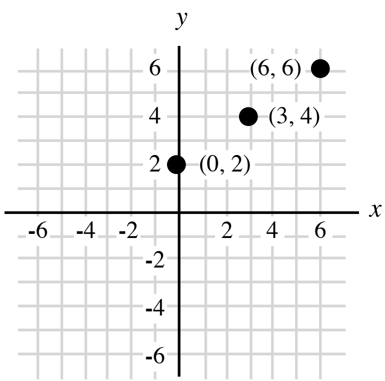
Appendix A.6 Functions

Relations: Domain and Range

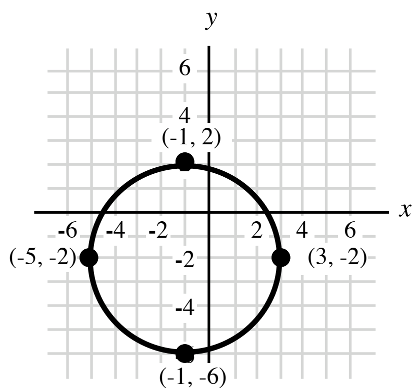
A **relation** is a set of ordered pairs. A relation can be a simple set of just a few ordered pairs, such as {(0, 2), (1, 3), (2, 4)}, or it can be infinite, such as the set of all points ton a line or a curve.

The **domain** of a relation is the set of all (possible) x-values, and the **range** is the set of all y-values.



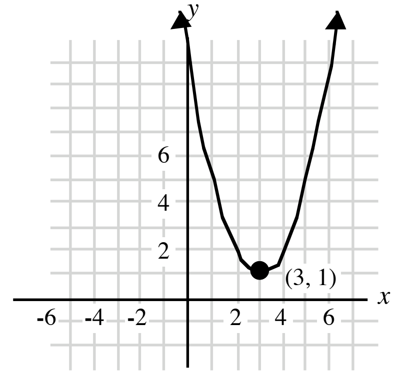
Domain: {0, 3, 6}

Range: {2, 4, 6}



Domain: [-5, 3]

Range: [-6, 2]



Domain: (-∞, ∞)*

Range: [1, ∞)

The domain and range are sets of numbers and we can represent each in one of several ways. In this appendix, you might come across any of these solution sets:

Words	Interval Notation	Set Builder Notation	Symbolically
all real numbers	$(-\infty, \infty)$	$\{ x \mid x$ is a real number $\}$	$\mathbb{R}$
x is between -5 and 3, inclusive	$[-5, 3]$	$\{ x \mid -5 \le x \le 3 \}$	$-5 \le x \le 3$
x is between -5 and 3, exclusive	$(-5, 3)$	$\{ x \mid -5 < x < 3 \}$	$-5 < x < 3$
x is greater than or equal to -1	$[-1, \infty)$	$\{ x \mid x \ge -1 \}$	$x \ge -1$
x is less than 2	$(-\infty, 2)$	$\{ x \mid x < 2 \}$	$x < 2$
x is not 7	$(-\infty, 7) \cup (7, \infty)$	$\{ x \mid x \ne 7 \}$	$\mathbb{R} - \{ 7 \}$

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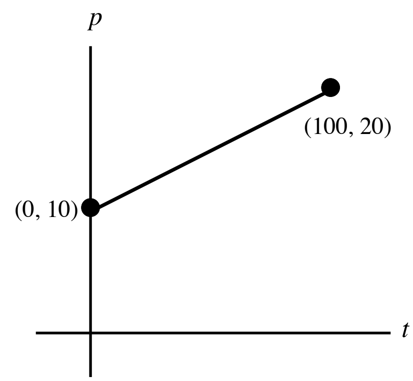
Some domains, called *restricted domains*, are truncated for one reason or another.

For example, the equation

$$p = 0.1t + 10$$

is a linear equation that represents the price, p dollars, of a cell phone plan, based on the number of text messages sent, t, (up to 100).

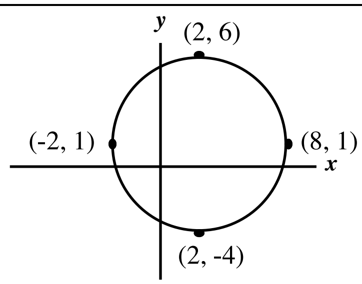


In this case, t cannot be negative, and values of t that are more than 100 do not add to the price (and are not considered here). So, ...

Domain: [0, 100] and Range: [10, 20]

You Try It 1 For each relation, identify both the domain and the range.

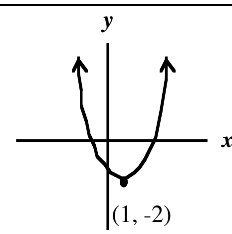
a)



Domain:

Range:

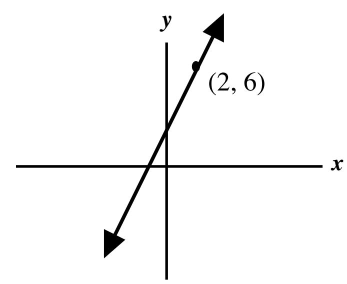
b)



Domain:

Range:

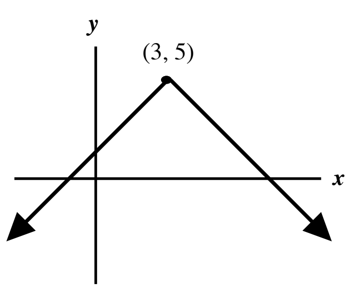
c)



Domain:

Range:

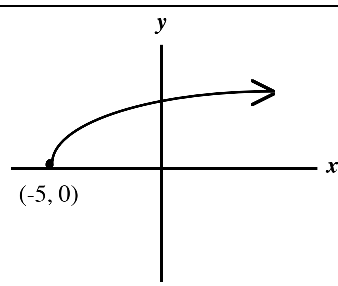
d)



Domain:

Range:

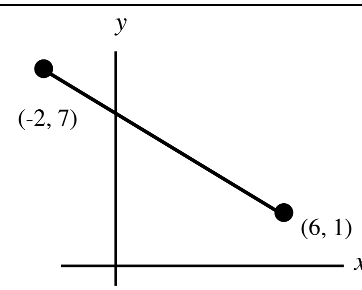
e)



Domain:

Range:

f)



Domain:

Range:

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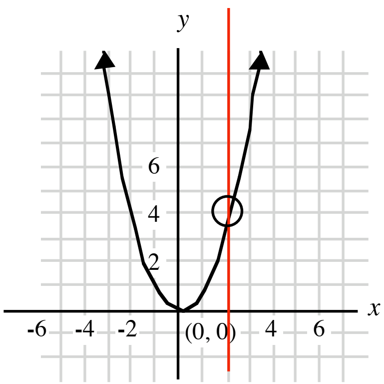
FUNCTION DEFINITION

A **function** is a relation such that, for every *x* there is only one *y*.

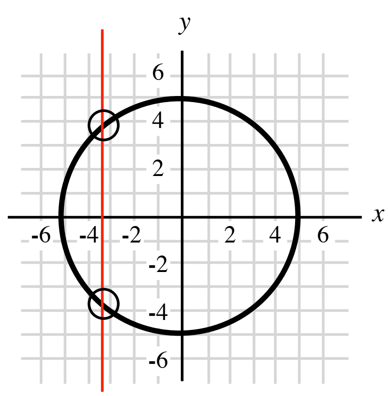
> A more formal definition of *function* is: for every element, x_i, of the domain of *f*, there is exactly one element, y_i, in the range of *f* to which x_i is assigned.

The *vertical line test* is a way to visually identify whether the graph of a relation is a function. If any vertical line can cross the graph in more than one place, then the graph is *not* a function; otherwise, it is a function.

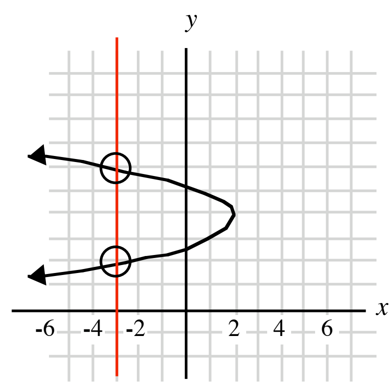
The basic idea behind the vertical line test is that we are visually checking each and every *x*-value in the domain, making sure that it corresponds to only one *y*-value.



This relation is a function.



This relation is *not* a function.



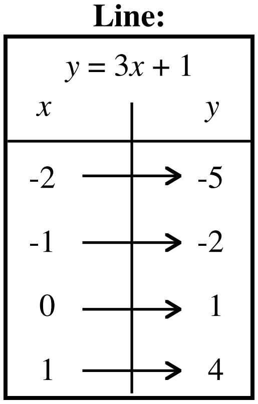
This relation is *not* a function.

ONE-TO-ONE FUNCTIONS

> A function is **one-to-one** if, for every *y* there is only one *x* that corresponds to it.
>
> A function that is not one-to-one is called **many-to-one**.

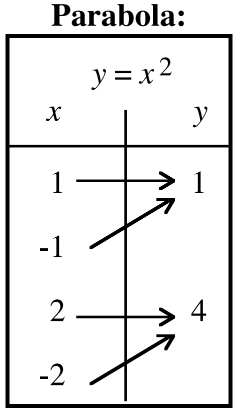
For example, a slanted line is one-to-one, but a parabola is not. Below are three tables showing the correspondence between domain and range values of a line, a parabola, and a sine wave.

Line:



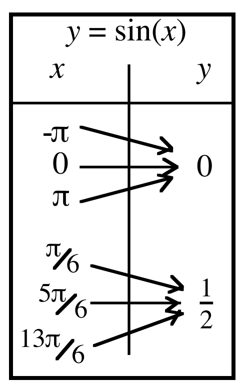
This function is one-to-one.

Parabola:



This function is many-to-one.

Sine wave:



This function is many-to-one.

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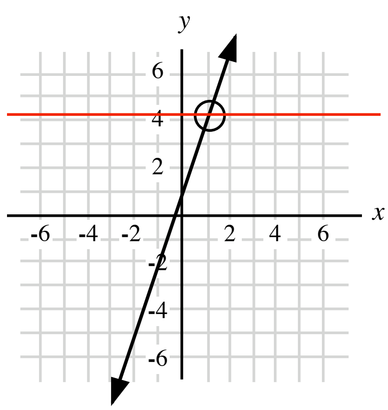
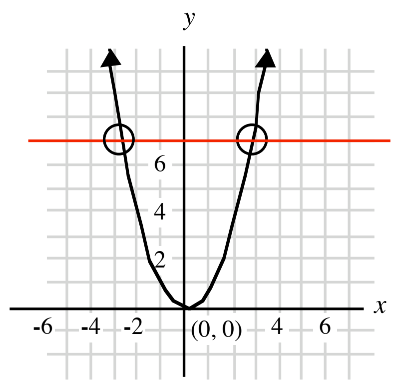
The *horizontal line test* is a way to visually identify whether the graph of a function is one-to-one or many-to-one. If any horizontal line can cross the graph in more than one place, then the function is many-to-one; otherwise, it is one-to-one.

$y = 3x + 1$



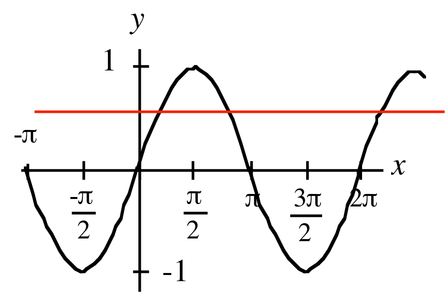
This function is one-to-one.

$y = x^2$



This function is many-to-many.

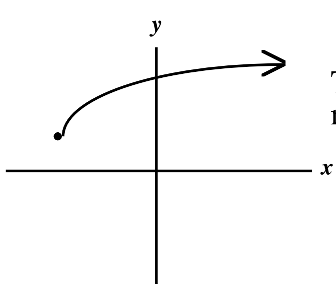
$y = \sin x$



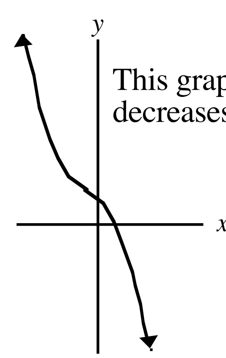
This function is many-to-one.

If a function has any "turn-around points" (such as the vertex of a parabola), then the function is many-to-one. The bottom line is, a function is <u>one-to-one</u> if it is either ...

all increasing: **or** **all decreasing:**



This graph continually rises from its vertex.



This graph continually decreases (from left to right).

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The Algebra of Functions

RESTRICTIONS ON THE DOMAIN

Some functions have natural restrictions. In particular,

(i) a denominator can never be 0 (The numerator is unaffected by this restriction.)

(ii) the radicand of a square root can never be negative.

We do not consider the option of imaginary numbers because the x- and y-axes are real number axes.

Example 1: Identify the domain of each function.

a) $y = \sqrt{2x + 1}$ b) $y = \dfrac{x + 2}{3x - 4}$ c) $y = x^2 + 4x - 1$

Procedure: Identify whether the function has a natural domain restriction or otherwise.

Answer:

a) The radicand cannot be negative:

$2x + 1 \geq 0$

$2x \geq -1$

Domain: $x \geq -\frac{1}{2}$

b) The denominator cannot be zero:

$3x - 4 \neq 0$

$3x \neq 4$

Domain: $x \neq \frac{4}{3}$

c) For polynomial functions, the domain is <u>all real numbers</u> (unless it has a given domain restriction).

Domain: $\mathbb{R}$

Note: For some functions, the range is not intuitive and is often found only after the function has been graphed.

<u>You Try It 2</u> For each function, identify the domain. (You are not asked to find the range.)

a) $y = \dfrac{2x - 9}{3 - 6x}$ b) $y = -x^3 + x - 2$ c) $y = \sqrt{4 - 5x}$

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FUNCTIONAL VALUES

Definitions:

1. The symbol $f(x)$ means the "function of x" and is called the **function notation**.

2. In function notation, the value within the parentheses is called the **argument** of the function.

3. The variable in an argument can be replaced by a number, called a **replacement value**. A replacement value must be an element (member) of the domain.

Caution: The parentheses used in the function notation, $f(x)$, do *not* mean "multiply."

When a replacement value is used, each and every occurrence of the variable gets replaced by this value.

For example, if $f(x) = 2x^2 - 5x + 4$, then

a) $f(3) = 2(3)^2 - 5(3) + 4$
$= 2(9) - 15 + 4$
$= 18 - 15 + 4$
$= 7$

b) $f(-4) = 2(-4)^2 - 5(-4) + 4$
$= 2(16) + 20 + 4$
$= 32 + 20 + 4$
$= 56$

You Try It 3 Given $f(x) = \sqrt{3x + 15}$, find the following:

a) $f(22)$ b) $f\left(\frac{1}{3}\right)$ c) $f(-2)$

You Try It 4 Given $g(x) = x^2 - 4x - 12$, find the following:

a) $g(0)$ b) $g(5)$ c) $g(-2)$

You Try It 5 Given $h(x) = -x^2 + 5$, find the following:

a) $h(4)$ b) $h\left(\frac{5}{2}\right)$ d) $h(-3)$

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We can also replace the argument with other variable arguments. For example, if

Example 2: Given $g(x) = x^2 - 3x - 4$, find the following:

a) $g(a)$ b) $g(2a)$ c) $g(a + 1)$

Procedure: For each, replace x with the requested argument.

Answer:

a) Replace each x with a:

$$\boxed{g(a) = a^2 - 3a - 4}$$

b) Replace each x with $2a$; simplify.

$g(2a) = (2a)^2 - 3(2a) - 4$

$$\boxed{g(2a) = 4a^2 - 6a - 4}$$

c) Replace each x with the quantity $(a + 1)$:

$g(a + 1) = (a + 1)^2 - 3(a + 1) - 4$

$= a^2 + 2a + 1 - 3a - 3 - 4$

$$\boxed{g(a + 1) = a^2 - a - 6}$$

You Try It 6 Given $h(x) = \dfrac{x^2 - 4}{x - 1}$, find the following:

a) $h(a^2)$ b) $h(a - 2)$

Composite Functions

We can also replace an argument in one function, $f(x)$, with another function, $g(x)$. This is called the *composition* of two functions. The result is another function.

$f(x)$ composed with $g(x)$ is written $f \circ g\,(x) = f[g(x)]$

$g(x)$ composed with $f(x)$ is written $g \circ f\,(x) = g\,[f(x)]$

It is even possible to consider a function composed with itself: $f \circ f\,(x) = f[f(x)]$

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Example 3: Given $f(x) = 3x - 1$ and $g(x) = x^2 + 2$, find the following:

a) $f \circ g\,(x)$ b) $g \circ f\,(x)$ c) $f \circ f\,(x)$

Procedure: For each, the "replacement value" is a full function.

Answer:

a) $f \circ g\,(x)$

$$\begin{aligned} f[g(x)] &= f(x^2 + 2) \\ &= 3(x^2 + 2) - 1 \\ &= 3x^2 + 6 - 1 \\ &= 3x^2 + 5 \end{aligned}$$

b) $g \circ f\,(x)$

$$\begin{aligned} g[f(x)] &= g(3x - 1) \\ &= (3x - 1)^2 + 2 \\ &= 9x^2 - 6x + 1 + 2 \\ &= 9x^2 - 6x + 3 \end{aligned}$$

c) $f \circ f\,(x)$

$$\begin{aligned} f[g(x)] &= f(3x - 1) \\ &= 3(3x - 1) - 1 \\ &= 9x - 3 - 1 \\ &= 9x - 4 \end{aligned}$$

You Try It 7 Given $f(x) = 3x - 2$ and $g(x) = x^2 - x$, find the following:

a) $f \circ g\,(x)$ b) $g \circ f\,(x)$

You Try It 8 Given $f(x) = \frac{2}{3}x - 1$ and $g(x) = \frac{3x + 3}{2}$, find the following:

a) $f \circ g\,(x)$ b) $g \circ f\,(x)$

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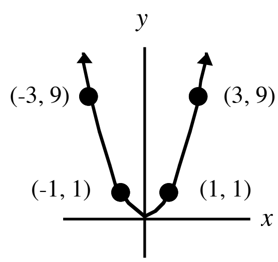
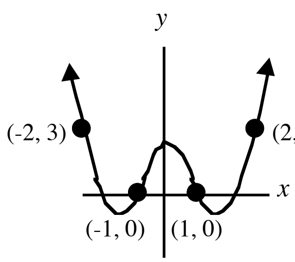
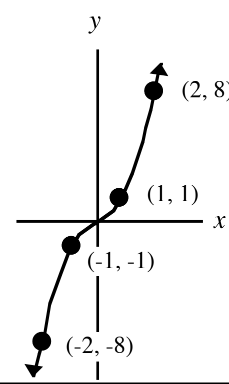
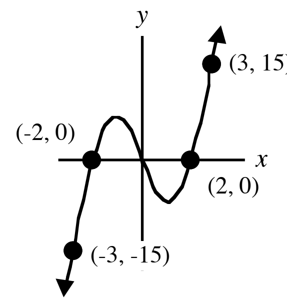
EVEN AND ODD FUNCTIONS

Some functions, not all, have the property of being either "odd" or "even" based on these criteria:

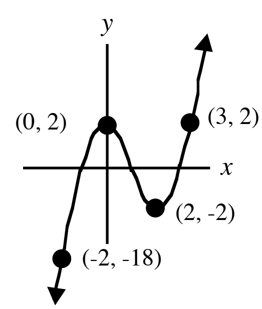
Function Type	**Algebraic Property**	**Graph Symmetry**
Even	$f(-x) = f(x)$	about the ***y*-axis**
Odd	$f(-x) = -f(x)$	about the **origin**

Examples of the graphs of even and odd functions:

Even	**Even**	**Odd**	**Odd**
$f(x) = x^2$	$g(x) = x^4 - 4x^2 + 3$	$h(x) = x^3$	$k(x) = x^3 - 4x$
			

At right is a function, $p(x)$, that is neither even nor odd; it has symmetry—about the point (1, 0)—but not about the origin or the y-axis ⟶

$p(x) = x^3 - 3x^2 + 2$



You might have noticed the examples of even functions have only even powers. Certainly $f(x) = x^2$ has an even power. So, too, does $g(x)$, which can be written $g(x) = x^4 - 4x^2 + 3x^0$.

Similarly, odd functions have only odd powers, such as $h(x) = x^3$ and $k(x) = x^3 - 4x^1$. Because $p(x) = x^3 - 3x^2 + 2$ has a mix of odd and even powers, it is neither odd nor even.

In-Class Example 4: What type of function is it that has symmetry about the x-axis? *(Hint: Draw a diagram.)*

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Here, again, are the *algebraic* properties of even and odd functions:

A function, $f(x)$, is **even** if $f(-x) = f(x)$ for all domain values.

A function, $f(x)$, is **odd** if $f(-x) = -f(x)$ for all domain values.

Example 5: Determine whether the function is even, odd, or neither.

a) $f(x) = x^3 - 4x$ b) $g(x) = 2x^2 + 6x - 8$ c) $h(x) = (x^3 + x)^2$

Procedure: Find $f(-x)$ and simplify. If the resulting function is the same as $f(x)$, then the function is *even*; if it is the opposite of $f(x)$, then it is *odd*; it might also be *neither* of these options.

Answer: **a)** $f(x) = x^3 - 4x$ Replace each x in the function with $-x$ and simplify.

$f(-x) = (-x)^3 - 4(-x)$ Evaluate each term.

$= -x^3 + 4x$ Because the lead term is negative, factor out -1.

$= -1(x^3 - 4x)$ In the parentheses is the original $f(x)$, so ...

$f(-x) = -f(x)$ **$f(x)$ is an odd function.**

b) $g(x) = 2x^2 + 6x - 8$ Replace each x in the function with $-x$ and simplify.

$g(-x) = 2(-x)^2 + 6(-x) - 8$ Evaluate each term.

$= 2x^2 - 6x - 8$ The lead term is the same as it is for $g(x)$, so g is not odd; however, the middle terms of $g(x)$ and $g(-x)$ are different, so ...

$g(-x) \neq -g(x)$ **$g(x)$ is neither even nor odd.**

c) $h(x) = (x^3 + x)^2$ Replace each x in the function with $-x$ and simplify.

$h(-x) = [(-x)^3 + (-x)]^2$ Evaluate each term inside the brackets.

$= [-x^3 - x]^2$ Inside the brackets, the lead term is negative, and we can factor out -1 from those two terms.

$= [-1(x^3 + x)]^2$ Use the distributive rule of exponents, $(ab)^n = a^n b^n$.

$= (-1)^2 \cdot (x^3 + x)^2$ Simplify: $(-1)^2 = +1$.

$= (x^3 + x)^2$ This is the same as $h(x)$, so ...

$h(-x) = h(x)$ **$h(x)$ is an even function.**

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In part c), $h(x) = (x^3 + x)^2$ is a bit confusing. It appears as though $h(x)$ might be an odd function because of the odd power on each x. However, because the quantity is being squared, it puts a twist on the even/odd notion of exponents, so maybe it's *neither*. In part c), we learn that $h(x)$ is actually an even function. Here is another technique that could be used:

In-Class Example 6: Show that $h(x) = (x^3 + x)^2$ is an even function by first squaring out the quantity.

You Try It Answers

YTI 1:	a)	Domain: $-2 \le x \le 8$ Range: $-4 \le y \le 6$	b)	Domain: $\mathbb{R}$ Range: $y \ge -2$	c)	Domain: $\mathbb{R}$ Range: $\mathbb{R}$
	d)	Domain: $\mathbb{R}$ Range: $y \le 5$	e)	Domain: $x \ge -5$ Range: $y \ge 0$	c)	Domain: $-2 \le x \le 6$ Range: $1 \le y \le 7$
YTI 2:	a)	Domain: $x \ne \frac{1}{2}$	b)	Domain: $\mathbb{R}$	c)	Domain: $x \le \frac{4}{5}$
YTI 3:	a)	9	b)	4	c)	3
YTI 4:	a)	-12	b)	-7	c)	0
YTI 5:	a)	-11	b)	$-\frac{5}{4}$	c)	-4

YTI 6: a) $h(a^2) = \dfrac{a^4 - 4}{a^2 - 1}$ b) $h(a - 2) = \dfrac{a^2 - 4a}{a - 3}$

YTI 7: a) $f \circ g\,(x) = 3x^2 - 3x - 2$ b) $g \circ f\,(x) = 9x^2 - 15x + 6$

YTI 8: a) $f \circ g\,(x) = x$ b) $g \circ f\,(x) = x$

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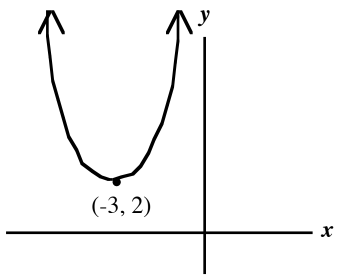
Focus Exercises

Given the graph of f(x), determine its domain and range.

1.



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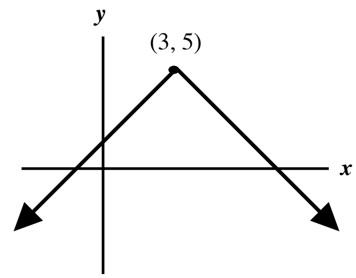

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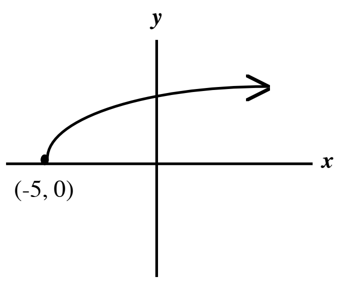

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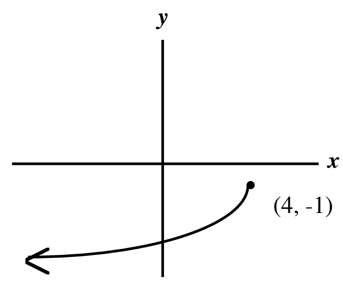

Identify the domain of the function. Keep in mind any possible restriction the domain may have.

5. $f(x) = \sqrt{2x - 6}$

6. $h(x) = \dfrac{x + 1}{3x - 5}$

7. $g(x) = \dfrac{2}{3}\,x - 4$

8. $f(x) = \sqrt{8 - 4x}$

9. $k(x) = \dfrac{x}{x^2 - 4}$

10. $f(x) = x^2 + 1$

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Find the requested functional value for $f(x) = \sqrt{2x + 5}$ and $g(x) = x^2 - x + 2$

11. $f(2)$

12. $f(10)$

13. $f\left(-\frac{1}{2}\right)$

14. $f(-2)$

15. $g(0)$

16. $g(5)$

17. $g(-1)$

18. $g(-3)$

Given $f(x) = 2x - 3$ and $g(x) = x^2 - x + 2$, find the following.

19. $f(3w)$

20. $g(2c)$

21. $g(x - 2)$

22. $f(x - 2)$

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23. $f \circ g\,(x)$

24. $g \circ f\,(x)$

For each, use the algebraic properties to determine whether the function is even, odd, or neither.

25. $f(x) = \frac{-2}{3}x$

26. $g(x) = 3x^2 - 5$

27. $h(x) = \sqrt[3]{x}$

28. $f(x) = x^3 + x + 1$

29. $g(x) = 4 - x^2$

30. $h(x) = (x^3 + 1)^2$

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