Appendix A.7 Inverse Functions

INTRODUCTION

The practice of Mathematics follows Newton's Third Law that says, "For every action, there is an equal and opposite reaction." The "equal and opposite reaction" is called *inverse*. For example, addition and subtraction are inverse operations; likewise, multiplication and division are inverse operations.

The inverse of a functions is simply called the *inverse function*, and we explore its meaning and derivation in this section.

IDENTITY VALUES IN MATHEMATICS

To understand the inverse of a function, it is important to first understand the notion of the *identity value* of an operation.

Note: An identity value is also called the *identity element*, or just, *identity*.

An **identity value** is best explained by example:

The identity value for addition—the *additive identity*—is 0 (zero). Adding 0 to any number creates no new value, as in 3 + 0 = 3.

Note: The identity value for multiplication, 1, is called the *multiplicative identity*.

In essence, an identity value is one which creates no change for a given operation.

Here is a table of common operations and their identity values:

Operation	Identity Value	Comparison Value	Applying the Identity Value
Addition	0	a	$a + 0 = a$ and $0 + a = a$
Multiplication	1	b	$b \cdot 1 = b$ and $1 \cdot b = b$
Function Composition	$I(x) = x$ *the identity function*	$f(x)$	$f\left[I(x)\right] = f(x)$

INVERSE VALUE

The *inverse value* of a number in an operation has a reversing—undoing—effect, resulting in the identity value of the operation.

An **inverse value** is best explained by example:

The inverse value for addition—the *additive inverse*—is called the *opposite*. For 3, the additive inverse is -3; for -7, the additive inverse is +7. When added, a number and its opposite always result in 0, the additive identity.

$$3 + (-3) = 0 \qquad\qquad -7 + 7 = 0$$

In essence, for a given operation, applying a value to its inverse results in the identity value for that operation.

Here is a table of common operations and their inverse values:

Operation	**Name of Inverse**	**Value of Inverse**	**Applying the Inverse Value**
Addition	*opposite*	$-a$ or $-1 \cdot a$	$a + (-a) = 0$ and $-a + a = 0$ 0 is the additive identity.
Multiplication	*reciprocal*	$b^{-1} = \frac{1}{b}$	$b \cdot \frac{1}{b} = 1$ and $\frac{1}{b} \cdot b = 1$ 1 is the multiplicative identity.
Function Composition	*The inverse of f(x)*	$f^{-1}(x)$	$f[f^{-1}(x)] = x = I(x)$ and $f^{-1}[f(x)] = x = I(x)$ $I(x)$ is the identity function.

There is a common thread that runs through the general notion of inverse, which is -1. Essentially, -1 means *inverse*. This idea shows up in a variety of ways, depending on the operation:

- for addition, the inverse of a number is -1 times the number: $-8 = -1 \cdot 8$
- for multiplication, the exponent -1 indicates the multiplicative inverse: $3^{-1} = \frac{1}{3}$
- for functions, the inverse of a function is written $f^{-1}(x)$.

Note: The -1 in the inverse function does *not* mean reciprocal.

In other words, $f^{-1}(x) \neq \frac{1}{f(x)}$

INVERSE FUNCTION COMPOSITION

To this point in the discussion, the most important property to know about inverse functions is

$$f[f^{-1}(x)] = x$$
$$\text{and} \quad f^{-1}[f(x)] = x$$

This also means that, if both $f[g(x)] = x$ and $g[f(x)] = x$, then f and g are inverses of each other. In other words, $g(x) = f^{-1}(x)$.

Under most circumstances, if we can demonstrate just one of these is true, either $f\,[g(x)] = x$ or $g[f(x)] = x$, it is sufficient for us to conclude $g(x) = f^{-1}(x)$. We could also say $f(x) = g^{-1}(x)$.

Example 1: For each pair of functions, use $f\,[g(x)]$ to show that $g(x) = f^{-1}(x)$.

a) $f(x) = \frac{2}{3}x - 1$ and $g(x) = \dfrac{3x + 3}{2}$

b) $f(x) = \sqrt[3]{5x + 2}$ and $g(x) = \frac{1}{5}x^3 - \frac{2}{5}$

Procedure: Show that $f\,[g(x)] = x$.

Answer: a) $f\,[g(x)]$ — Place $\frac{3x+3}{2}$ as the argument of f.

$= f\left[\dfrac{3x+3}{2}\right]$ — Place this argument into $f(x)$.

$= \frac{2}{3}\left[\dfrac{3x+3}{2}\right] - 1$ — Divide out the 2's.

$= \frac{1}{3}(3x + 3) - 1$ — Distribute $\frac{1}{3}$.

$= x + 1 - 1$ — Combine like terms: $1 - 1 = 0$.

$= x$ — **So, $g(x) = f^{-1}(x)$.**

b) $f\,[g(x)]$ — Place $\frac{1}{5}x^3 - \frac{2}{5}$ as the argument of f.

$= f\left[\frac{1}{5}x^3 - \frac{2}{5}\right]$ — Place this argument into $f(x)$.

$= \sqrt[3]{5\left(\frac{1}{5}x^3 - \frac{2}{5}\right) + 2}$ — Distribute 5.

$= \sqrt[3]{x^3 - 2 + 2}$ — Combine like terms: $-2 + 2 = 0$.

$= \sqrt[3]{x^3}$ — Simplify.

$= x$ — **So, $g(x) = f^{-1}(x)$.**

In-Class Example 2: For each pair of functions, use $g\,[f(x)]$ to show that $g(x) = f^{-1}(x)$.
(These are the same functions as in Example 1. Here, you are asked to use the other composition.)

a) $f(x) = \frac{2}{3}x - 1$ and $g(x) = \frac{3x + 3}{2}$

$g[f(x)] =$

b) $f(x) = \sqrt[3]{5x + 2}$ and $g(x) = \frac{1}{5}x^3 - \frac{2}{5}$

$g[f(x)] =$

INVERSE FUNCTIONS: DOMAIN AND RANGE

In general, a function and its inverse interchange x- and y-values: the domain of f is the range of f^{-1} and vice-versa. Consider the following ordered pairs (points) on $f(x)$ and $f^{-1}(x)$ from Example 1:

$f(x) = \frac{2}{3}x - 1$

Domain x		Range y
-3	→	-3
0	→	-1
3	→	1
6	→	3

$f^{-1}(x) = \frac{3x + 3}{2}$

Domain x		Range y
-3	→	-3
-1	→	0
1	→	3
3	→	6

What this suggests is, if $f(a) = b$, then $f^{-1}(b) = a$. In other words, if the point (a, b) is on $f(x)$, then the point (b, a) is on $f^{-1}(x)$.

For example, $f(6) = 3$ and $f^{-1}(3) = 6$.

In-Class Example 3: Consider this unspecified function, $f(x)$. Fill in the table for $f^{-1}(x)$:

$f(x)$ Domain		$f(x)$ Range	$f^{-1}(x)$ Domain		$f^{-1}(x)$ Range
-5	→	10		→	
-1	→	4		→	
7	→	0		→	
12	→	-2		→	

You Try It 1 Given the set of ordered pairs of $f(x)$, write the set of ordered pairs for $f^{-1}(x)$.

$f(x) = \{(1, 10), (4, 9), (0, 5), (3, 3), (-2, -6)\}$

$f^{-1}(x) = \{ \qquad \}$

Does this idea of interchanging x- and y-values work for all functions? To answer this question, let's look at some ordered pairs of a familiar function, $f(x) = x^2$:

$f(x) = x^2$: x		y	$f^{-1}(x)$??: x		y
0	→	0	0	→	0
1	→	1	1	→	1
-1	↗			↘	-1
2	→	4	4	→	2
-2	↗			↘	-2
3	→	9	9	→	3
-3	↗			↘	-3

In the second set of values (the inverse), we see that the relation is actually not a function so there is no $f^{-1}(x)$. This happens because $f(x)$ is not a one-to-one function. This means ...

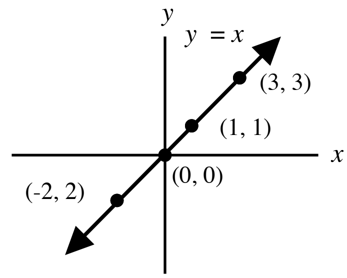
> A function, $f(x)$, has an inverse, $f^{-1}(x)$, if and only if $f(x)$ is **one-to-one**.

Because a one-to-one function passes both the horizontal and vertical line tests, when we interchange the x- and y-values, we create a new one-to-one function.

UNDERSTANDING THE INVERSE FUNCTION GRAPHICALLY

To understand the graphical relationship between a function and its inverse, let's first consider the line $y = x$:

> $y = x$ passes through the origin and every point in which the x- and y-coordinates are identical:
>
> (1, 1), (2, 2), (5, 5), (-3, -3), and so on.



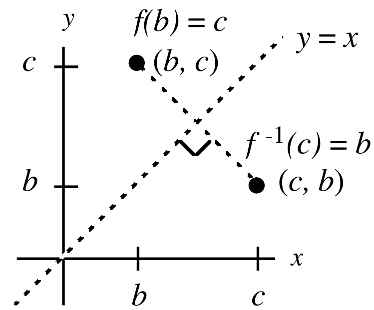
The importance of this line in the graphing of inverse functions is ...

> $y = x$ is the <u>line of symmetry</u> between the graphs of $f(x)$ and its inverse.

As stated earlier, if $f(b) = c$, then $f^{-1}(c) = b$.

This means the point (b, c) is on the graph of $f(x)$ and the point (c, b) is on the graph of $f^{-1}(x)$.

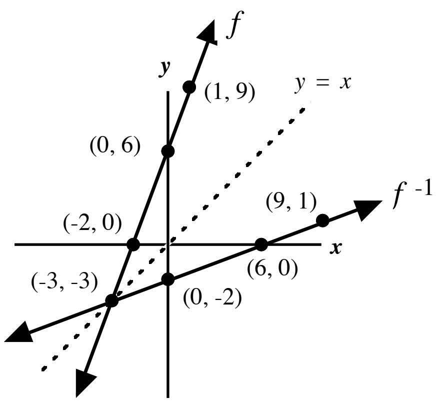
Notice that the two points are symmetric about the line $y = x$. Furthermore, this line of symmetry is a perpendicular bisector of the line segment that connects the two points.

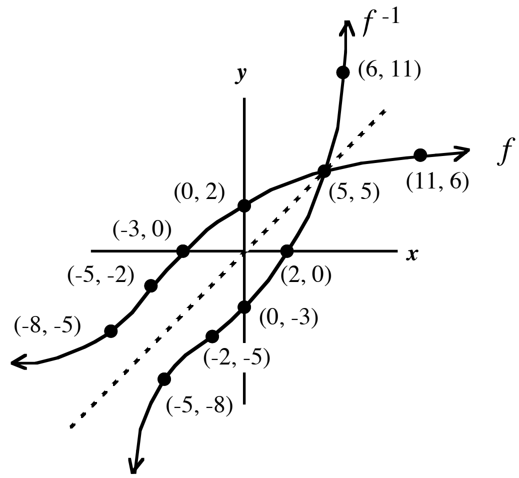


As composite functions: $f^{-1}[f(b)] = f^{-1}(c) = b \rightarrow f^{-1}[f(b)] = b$

and $f[f^{-1}(c)] = f(b) = c \rightarrow f[f^{-1}(c)] = c$

Each of these diagrams, below, shows the graphing of $f(x)$ and $f^{-1}(x)$. Notice the interchanging of x- and y-coordinates from f to f^{-1}, and the visual symmetry about the line $y = x$. Notice also that f and f^{-1} share a point whenever they cross the line $y = x$.





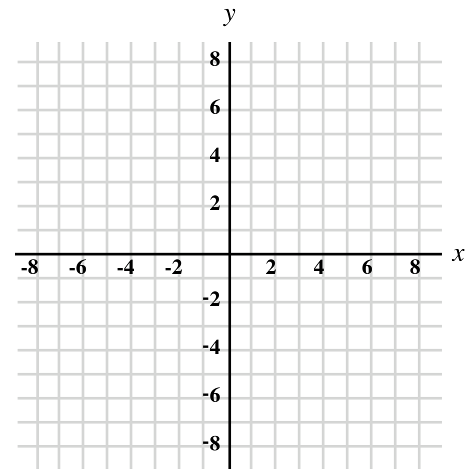
In-Class Example 4: Consider a smooth, connected graph of $f(x)$ that contains the points (5, 8), (4, 4), (2, 0), (0, -2), (-5, -5), and (-8, -6), among many others.

a) Plot these points in the x-y-plane and draw a smooth curve that passes through them.

b) Invert each ordered pair and write the new set of points. These points are part of $f^{-1}(x)$. Plot each ordered pair and draw a smooth curve that passes through them.

c) Lastly, draw the line $y = x$ as a dashed line.

d) Discuss the relationships between the graphs of $f(x)$, $f^{-1}(x)$, and $y = x$.

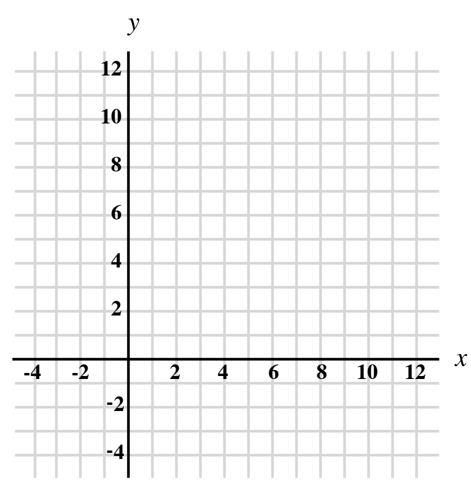


In-Class Example 5: Consider the graph of $f(x) = x^2$.

a) Draw the graph of $f(x)$ in the x-y-plane.

b) Invert the points on $f(x)$ and plot these points.

c) What do you notice about this second graph?



What you see is the inverse of the parabola does not pass the vertical line test; the inverse is a relation, but it is not a function. Does this mean that there is no $f^{-1}(x)$ for $f(x) = x^2$? Yes and no.

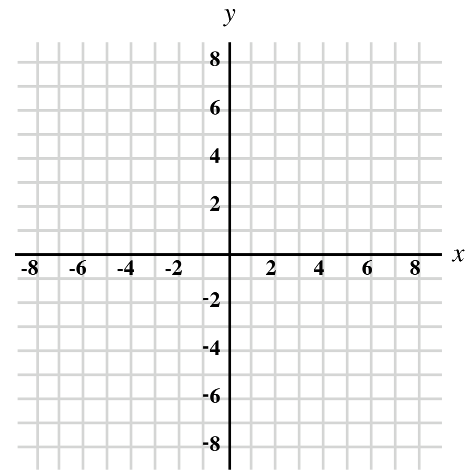
In its full form, $f(x) = x^2$ has no inverse function. However, we can restrict the domain to $x \geq 0$. This excludes negative values of x, so the graph is only increasing and not decreasing.

In-Class Example 6: Now consider the graph of $f(x) = x^2$, $x \geq 0$.

a) Draw the graph of $f(x)$ in the x-y-plane.

b) Invert the points on $f(x)$ and plot these points.

c) What do you notice about this second graph?



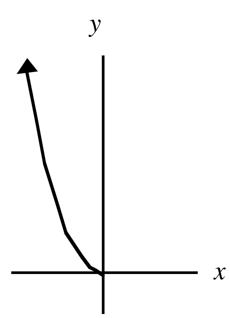
What we learn from this example is ...

> A function, f, must first be one-to-one for there to be an inverse function, f^{-1}.

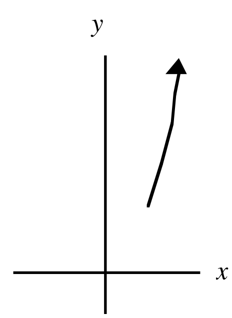
If a function is not already one-to-one, then to create a functional inverse, we must restrict the domain of f to be an interval where the graph is one-to-one. On this interval, the graph must be all increasing or all decreasing.

For $f(x) = x^2$, there are many interval options where we can restrict the domain and make the graph one-to-one. Here are a few example,

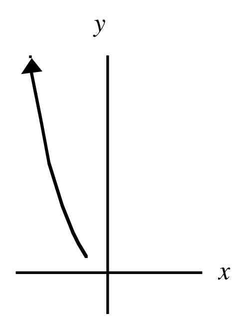
The left side only: $x \leq 0$



Quadrant I only: $x \geq 2$



Quadrant II only: $x \leq -1$



THE ALGEBRA OF AN INVERSE FUNCTION

Given a one-to-one function, $f(x)$, we can often find $f^{-1}(x)$ by using this step-by-step process:

Given $f(x) = 3x - 5$; find $f^{-1}(x)$.

1. Introduce y into the equation by replacing $f(x)$ with y, as shown at right.

2. Introduce $f^{-1}(x)$. by interchanging x and y in the equation. At this point, we are no longer in f, but in the new function, f^{-1}.

3. Solve this new equation for y.

4. Replace y with $f^{-1}(x)$.

Start with $f(x) = 3x - 5$

1. $f(x)$: $y = 3x - 5$

2. $f^{-1}(x)$: $x = 3y - 5$

3. Add 5 to each side:

$$x + 5 = 3y$$

Divide each side by 3:

$$\frac{x+5}{3} = y$$

4. $\boxed{f^{-1}(x) = \frac{x+5}{3}}$

Example 7: Given $f(x) = \sqrt[3]{2x-7}$, find its inverse, $f^{-1}(x)$.

Procedure: Follow this step-by-step outline.

Answer: $f(x) = \sqrt[3]{2x-7}$

1. Introduce y into the equation $\quad f(x)$: $y = \sqrt[3]{2x-7}$

2. Introduce $f^{-1}(x)$; interchange x and y. $\quad f^{-1}(x)$: $x = \sqrt[3]{2y-7}$

3. Solve for y.

$(x)^3 = \left(\sqrt[3]{2y-7}\right)^3$ Cube each side.

$x^3 = 2y - 7$ Add 7 to each side.

$x^3 + 7 = 2y$ Divide each side by 2.

$$\frac{x^3+7}{2} = y$$

4. Write the answer as $f^{-1}(x)$. $\quad \boxed{f^{-1}(x) = \frac{x^3+7}{2}}$

To verify that we have found the inverse function correctly, we can show that both $f\left[f^{-1}(x)\right] = x$ and $f^{-1}\left[f(x)\right] = x$:

a) $f\left[f^{-1}(x)\right]$

$$= f\left[\frac{x^3+7}{2}\right]$$

$$= \sqrt[3]{2\left[\frac{x^3+7}{2}\right] - 7}$$

$$= \sqrt[3]{x^3+7-7}$$

$$= \sqrt[3]{x^3}$$

$= x$ ✓ Verified

b) $f^{-1}\left[f(x)\right]$

$$= f^{-1}\left[\sqrt[3]{2x-7}\;\right]$$

$$= \frac{\left(\sqrt[3]{2x-7}\right)^3 + 7}{2}$$

$$= \frac{2x-7+7}{2}$$

$$= \frac{2x}{2}$$

$= x$ ✓ Verified

Example 8: Given $f(x) = \dfrac{2x\ +\ 5}{x}$, find its inverse, $f^{-1}(x)$.

Procedure: Because $f(x)$ has more than one x value, when we interchange the x and y, we must replace each x (numerator and denominator) with y.

Answer: $f(x) = \dfrac{2x\ +\ 5}{x}$

1. Introduce y into the equation. $f(x)$: $y = \dfrac{2x\ +\ 5}{x}$

2. Introduce $f^{-1}(x)$; interchange x and y. $f^{-1}(x)$: $x = \dfrac{2y\ +\ 5}{y}$ Multiply each side by y.

3. Solve for y.

$y \cdot x = \dfrac{2y\ +\ 5}{y} \cdot \dfrac{y}{1}$ Simplify.

$yx = 2y + 5$ Add $-2y$ to each side.

$yx - 2y = 5$ Factor out y on the left side.

$y(x-2) = 5$ Divide each side by $(x-2)$.

$$y = \frac{5}{x-2}$$

4. Write the answer as $f^{-1}(x)$. $\boxed{f^{-1}(x) = \dfrac{5}{x-2}}$

SUMMARY OF INVERSE FUNCTIONS

Here is a summary of the essential features (attributes) of an inverse function:

1. A function, $f(x)$, has an inverse, $f^{-1}(x)$, if and only if $f(x)$ is one-to-one.

2. The domain of f is the range if f^{-1}, and the range of f is the domain if f^{-1}.

3. The graphs of f and f^{-1} are symmetric with each other about the line $y = x$.

4. To find $f^{-1}(x)$ from $f(x)$, follow these steps

a) Replace $f(x)$ with y to introduce y into the equation.

b) Interchange x and y.

c) Solve for y.

c) Replace y with $f^{-1}(x)$.

You Try It Answer

YTI 1: $f^{-1}(x) = \{(10, 1), (9, 4), (5, 0), (3, 3), (-6, -2)\}$

Focus Exercises

Given the function f as a set of ordered pairs. First, determine whether f is one-to-one, and if it is, write the set of ordered pairs for f^{-1}.

1. $f(x) = \{ (-3, 8), (2, 4), (5, 1), (4, -3), (1, 5) \}$

2. $f(x) = \{ (6, 8), (3, 2), (1, -1), (-1, 1), (-3, 2) \}$

Given: f is a one-to-one function and its inverse is f^{-1}.

3. If $f(2) = 5$, then what is $f^{-1}(5)$?

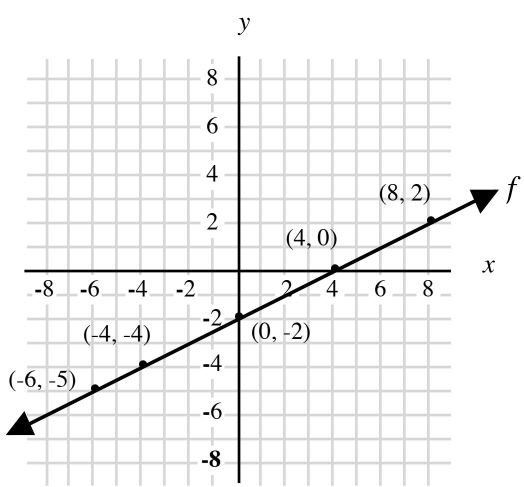
4. If $f(-3) = -7$, then what is $f^{-1}(-7)$?

5. If $f^{-1}(-3) = 1$, then what is $f(1)$?

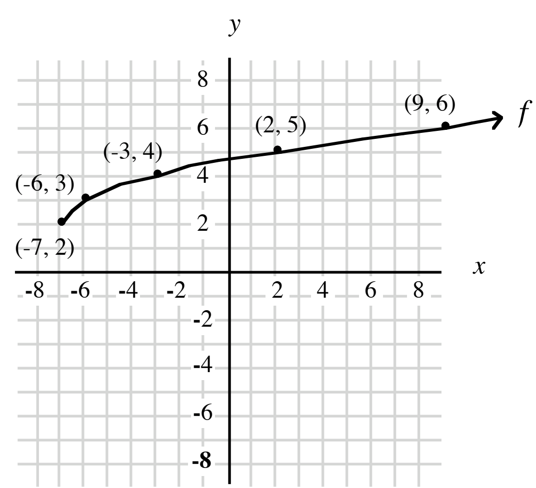
6. If $f^{-1}(0) = -4$, then what is $f(-4)$?

Given the graph of f(x), draw the graph of $f^{-1}(x)$.

7.



8.



Given the one-to-one function f(x), find $f^{-1}(x)$. *Verify the answer using either* $f\,[f^{-1}(x)]$ *or* $f^{-1}\,[f(x)]$.

9. $f(x) = 3x + 10$

Verify using either $f\,[f^{-1}(x)]$ *or* $f^{-1}\,[f(x)]$.

10. $f(x) = \dfrac{x - 3}{2}$

Verify using either $f\,[f^{-1}(x)]$ *or* $f^{-1}\,[f(x)]$.

11. $f(x) = \sqrt[3]{5x + 6}$ *Verify using either* $f\left[f^{-1}(x)\right]$ *or* $f^{-1}\left[f(x)\right]$.

12. $f(x) = \dfrac{2}{x - 4}$ *Verify using either* $f\left[f^{-1}(x)\right]$ *or* $f^{-1}\left[f(x)\right]$.