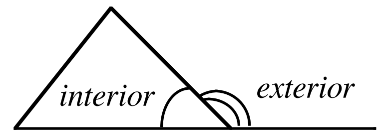
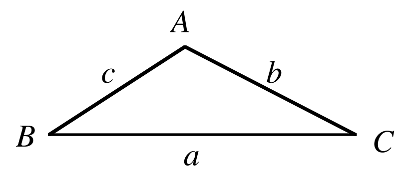
Section 1.3 Triangles

LABELING A TRIANGLE



The line segments that form a triangle are called the ***sides*** of the triangle. Each pair of sides forms an angle, called an ***interior*** angle, and each triangle has three interior angles.

If we extend one of the sides, it forms an angle outside of the triangle, called an **exterior** angle. An exterior angle is supplementary to its adjacent interior angle.



Each interior angle is also referred to as the ***included angle*** between the two sides that create it. For example, $\angle B$ is the included angle between sides $\overline{BC}$ and $\overline{BA}$.

Each point, *A*, *B* and *C*, is called a ***vertex***, and together they are called *vertices* (the plural of vertex).

Whereas the labeling of the vertices is usually arbitrary (*A*, *B* and *C* can go in any position around the triangle), labeling the sides as *a*, *b* and *c* is *not* arbitrary. For example, side *c* must go opposite $\angle C$, and so on.

A property of the sides of a triangle is that the sum of the lengths of any two sides is greater than the length of the third side.

$b + c > a$ and $a < b + c$

$a + c > b$ and $b < a + c$

$a + b > c$ and $c < a + b$

This is especially true for the longest side:

the length of the shortest side + the length of the middle side > the length of the longest side

Example 1: Can these be the lengths of the sides of a triangle? Assume all measures are in inches.

a) 5, 8, 10 b) 3, 4, 8 c) 5, 7, 12

Answer:

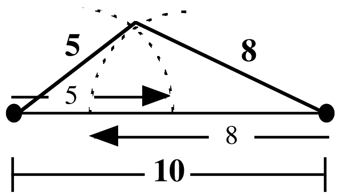
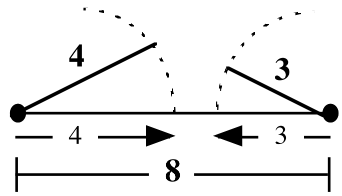
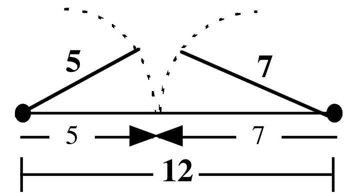
a) Yes because the sum of the lengths of the shorter legs, 5 + 8 = 13, is greater than the longest side, 10.

b) No, because 3 + 4 is not greater than 8.

c) No, because 5 + 7 is not *greater* than 12.

Note: A visual answer is on the next page.

Here is a visual answer for the three sets of side measures in Example 1:

a)	b)	c)
		
Yes, these measures form a triangle.	No, these measures do not form a triangle.	No, these measures do not form a triangle.

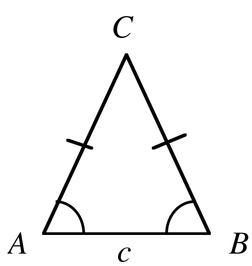
Starting with the longest side, draw it horizontally. Then, think of the other two side measures—starting at either end of the longest side—connected by a swinging hinge. If the two smaller sides can swing away from the longest side and meet somewhere, a triangle can be formed, as in a). Otherwise, no triangle can be formed, as in b) and c).

CLASSIFICATION OF TRIANGLES BY SIDE MEASURES

Isosceles Triangle:

i) Two sides are congruent to each other;

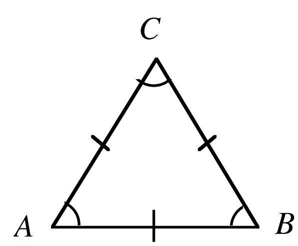
ii) the two angles, $\angle A$ and $\angle B$, opposite the congruent sides are called **base angles** and are congruent to each other.



Equilateral Triangle:

i) All three sides are congruent to each other;

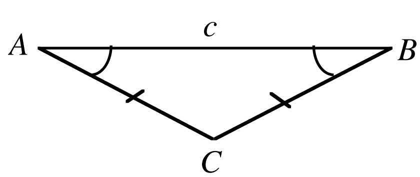
ii) all three angles are congruent to each other.



Is an equilateral triangle also isosceles?

Isosceles Triangle:

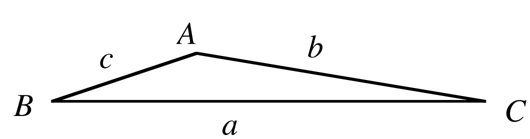
An isosceles triangle can be oriented differently than the one presented above. In this case, the congruent angles are still called base angles.



Scalene Triangle:

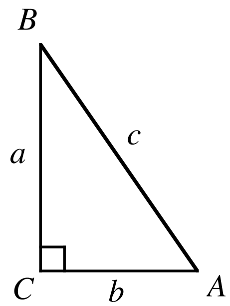
i) No two sides are congruent to each other;

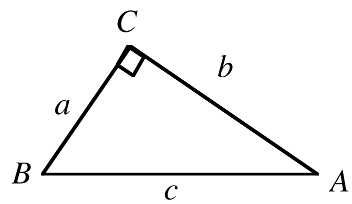
ii) likewise, no two angles are congruent to each other.

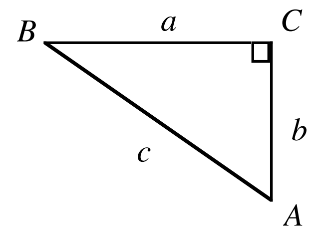


CLASSIFICATION OF TRIANGLES BY ANGLE MEASURES

Right Triangle: a triangle with one right angle. (It is not possible for a triangle to have two right angles.)

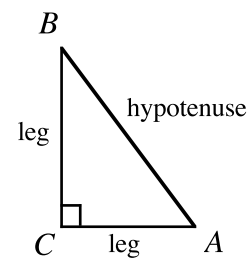




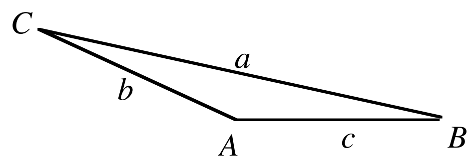


In a right triangle, the side opposite the right angle is called the ***hypotenuse***. The other two sides are called ***legs*** of the right triangle.

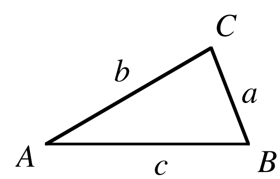
Only right triangles have hypotenuses, and the hypotenuse is always the longest side in a right triangle.



Obtuse Triangle: A triangle in which one angle is greater than 90°.

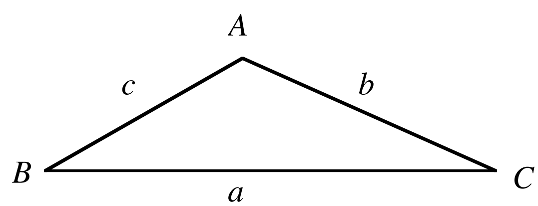


Acute Triangle: A triangle in which all three angles are less than 90°.



Oblique Triangle: A triangle in which no angle is a right angle.

Obtuse triangles and acute triangles are both examples of oblique triangles.

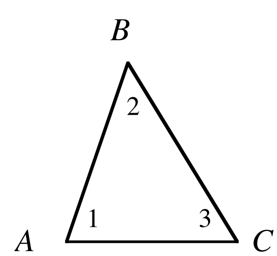


THE SUM OF THE ANGLES IN A TRIANGLE

> The sum of the interior angles in any triangle is 180°.

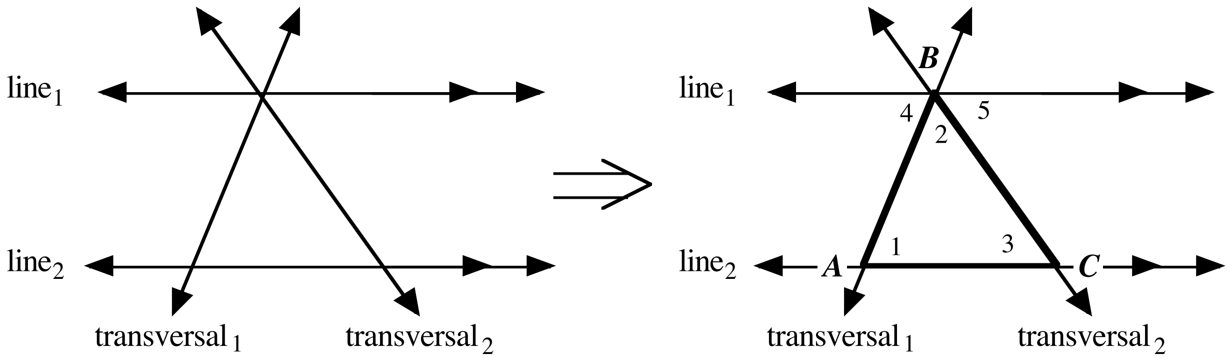
This means that $m\angle 1 + m\angle 2 + m\angle 3 = 180°$.

and that $m\angle A + m\angle B + m\angle C = 180°$.



Proof:

To show that the sum of the angles in a triangle is 180° we can use two parallel lines and two transversals:



At point B, three adjacent angles form line_1, so their sum must be that of a straight angle, 180°:

$$m\angle 4 + m\angle 2 + m\angle 5 = 180°$$

Also, $\angle 1$ and $\angle 4$ are alternate interior angles formed by the parallel lines and transversal_1, so

$$\angle 1 \cong \angle 4 \quad \text{which means} \quad m\angle 1 = m\angle 4.$$

Likewise, $\angle 3$ and $\angle 5$ are alternate interior angles formed by the parallel lines and transversal_2, so

$$\angle 3 \cong \angle 5 \quad \text{which means} \quad m\angle 3 = m\angle 5.$$

Replacing $\angle 4$ with $\angle 1$ and $\angle 5$ with $\angle 3$ we get

$$\boxed{m\angle 1 + m\angle 2 + m\angle 3 = 180°}$$

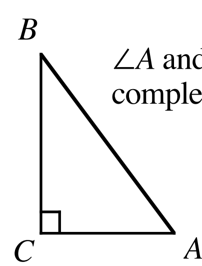
Hence, the sum of the (interior) angles in the triangle is 180°. This is true for every triangle.

THE ACUTE ANGLES IN A RIGHT TRIANGLE

In every triangle, at least two of the angles are acute. This is true for right triangles, obtuse triangles, and acute triangles.

In a right triangle, the two acute angles are complementary. This is easy to demonstrate:

$$m\angle C + m\angle A + m\angle B = 180°$$
$$90° + m\angle A + m\angle B = 180°$$
$$m\angle A + m\angle B = 90°$$



$\angle A$ and $\angle B$ are complementary.

So, $\angle A$ and $\angle B$ are complementary.

Example 2: Consider ΔABC. Given the measures of two of the angles, find the measure of the third angle.

a) $m\angle A = 25°$ and $m\angle B = 71°$. Find $m\angle C$.

b) $m\angle C = 103°\ 15'\ 42''$ and $m\angle A = 22°\ 48'\ 35''$. Find $m\angle B$.

Answer: Add the two known angles and subtract their sum from 180°.

a) $25° + 71° = 96°$; $m\angle C = 180° - 96° = 84°$

b)

$$\begin{array}{r} 103°\ 15'\ 42'' \\ +\ 22°\ 48'\ 35'' \\ \hline 125°\ 63'\ 77'' \end{array} \Rightarrow \underset{\textit{Adjust the seconds:}}{125°\ 64'\ 17''} \Rightarrow \underset{\textit{Adjust the minutes:}}{126°\ 04'\ 17''}$$

Make 1° into 60' → *Make 1' into 60"* → *Now subtract*

$$\begin{array}{r} 180°\ 00'\ 00'' \\ -\ 126°\ 04'\ 17'' \\ \hline \end{array} \Rightarrow \begin{array}{r} 179°\ 60'\ 00'' \\ -\ 126°\ 04'\ 17'' \\ \hline \end{array} \Rightarrow \begin{array}{r} 179°\ 59'\ 60'' \\ -\ 126°\ 04'\ 17'' \\ \hline \end{array}$$

$\boldsymbol{m\angle B} = 53°\ 55'\ 43''$

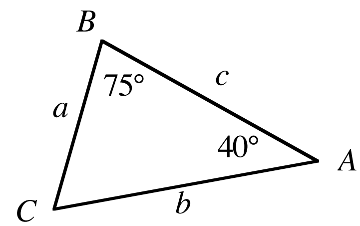
SIDES OPPOSITE ANGLES

In any triangle, <u>the shortest side is always opposite the smallest angle</u>. Likewise, <u>the longest side is always opposite the largest angle.</u>

This is why the angles in an equilateral triangle are all the same measure: there is no one side longer than another, so there is no angle larger than another.

Likewise, this is why the base angles of an isosceles triangle are congruent: the sides opposite the base angles are the same length, neither is longer nor shorter than the other, so the base angles must also be the same measure.

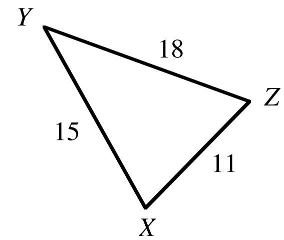
Example 3: List the sides in order from shortest to longest.



Procedure: First, $m\angle C = 180° - (75° + 40°) = 65°$.

Next, the shortest side is opposite the smallest angle, and so on.

Answer: The lengths of the sides from shortest to longest are $\boxed{a,\ c,\ b.}$

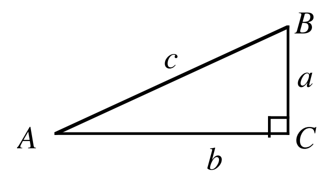


Example 4: List the angles in order from smallest to largest.

Procedure: The smallest angle is opposite the shortest side, and so on.

Answer: The size of the angles from smallest to largest are $\boxed{Y,\ Z,\ X.}$

THE PYTHAGOREAN THEOREM



In any right triangle ABC, with $m\angle C = 90°$, the lengths of the sides have the familiar Pythagorean relationship:

$$leg^2 + leg^2 = hypotenuse^2$$

$$a^2 + b^2 = c^2$$

Example 5: In a right triangle, given the measures of two sides, find the length of the third side. Here, c is the length of the hypotenuse and a and b are lengths of legs. Simplify.

a) $c = 13$ in. and $a = 12$ in. Find b.

b) $c = \sqrt{7}$ ft and $b = 2$ ft. Find a.

c) $a = \sqrt{5}$ km and $b = \sqrt{3}$ km. Find c.

d) $a = 3\sqrt{2}$ cm and $b = 4\sqrt{3}$ cm. Find c.

Answer: Use the Pythagorean Theorem to find the missing side value. Even though the solving of each equation leads to two solutions, one positive and one negative, we use only positive solutions for lengths and distances.

a)
$$\begin{aligned} 12^2 + b^2 &= 13^2 \\ 144 + b^2 &= 169 \\ b^2 &= 25 \\ b &= \pm\sqrt{25} \\ b &= 5 \text{ in.} \end{aligned}$$

b)
$$\begin{aligned} a^2 + 2^2 &= \left(\sqrt{7}\right)^2 \\ a^2 + 4 &= 7 \\ a^2 &= 3 \\ a &= \pm\sqrt{3} \\ a &= \sqrt{3} \text{ ft} \end{aligned}$$

c)
$$\begin{aligned} \left(\sqrt{5}\right)^2 + \left(\sqrt{3}\right)^2 &= c^2 \\ 5 + 3 &= c^2 \\ 8 &= c^2 \\ \pm\sqrt{8} &= c \\ c &= 2\sqrt{2} \text{ km} \end{aligned}$$

d)
$$\begin{aligned} \left(3\sqrt{2}\right)^2 + \left(4\sqrt{3}\right)^2 &= c^2 \\ 9\cdot 2 + 16\cdot 3 &= c^2 \\ 18 + 48 &= c^2 \\ 66 &= c^2 \\ \pm\sqrt{66} &= c \\ c &= \sqrt{66} \text{ cm} \end{aligned}$$

PYTHAGOREAN TRIPLES

Any set of three positive integers—a, b, and c—that have such a Pythagorean relationship is called a ***Pythagorean triple*** and can be written (a, b, c).

The set of numbers (3, 4, 5) is one such Pythagorean triple. As such, a triangle that has side measures (in inches or feet or meters) of 3, 4 and 5 is a *right triangle*. The longest measure is the length of the hypotenuse:

$$3^2 + 4^2 = 5^2$$

$$9 + 16 = 25 \quad \text{which is true!}$$

Note: It is true that $3, \sqrt{7}$ and 4 can be the sides of a right triangle, because $3^2 + (\sqrt{7})^2 = 4^2$. However, these numbers do *not* form a Pythagorean triple because not all of them are integers.

Generating Pythagorean Triples

For any two positive integers, m and n where $m > n$, we can generate a Pythagorean triple by applying the following formulas:

$$a = m^2 - n^2 \qquad b = 2mn \qquad c = m^2 + n^2$$

For example, if $m = 4$ and $n = 3$, then

$$a = 4^2 - 3^2 \qquad b = 2 \cdot 4 \cdot 3 \qquad c = 4^2 + 3^2$$

$$a = 16 - 9 \qquad b = 24 \qquad c = 16 + 9$$

$$a = 7 \qquad b = 24 \qquad c = 25$$

So, (7, 24, 25) is a Pythagorean triple: $7^2 + 24^2 = 25^2$

$$49 + 576 = 625$$

$$625 = 625 \quad \textit{True!}$$

Here is why these relationships generate Pythagorean triples:

$$a^2 + b^2 = c^2$$

$$(m^2 - n^2)^2 + (2mn)^2 \stackrel{?}{=} (m^2 + n^2)^2$$

$$m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \stackrel{?}{=} m^4 + 2m^2n^2 + n^4$$

$$m^4 - 2m^2n^2 + 4m^2n^2 + n^4 \stackrel{?}{=} m^4 + 2m^2n^2 + n^4$$

$$m^4 + 2m^2n^2 + n^4 = m^4 + 2m^2n^2 + n^4$$

Section 1.3 Focus Exercises

Can these be the lengths of the sides of a triangle? Assume all measures are in inches.

1. 16, 12, 8

2. 18, 32, 14

3. 61, 28, 32

4. 47.8, 16.8, 32.7

5. $4\sqrt{3}$, $5\sqrt{3}$, $2\sqrt{3}$

6. $\sqrt{18}$, $\sqrt{8}$, $\sqrt{32}$

In ΔABC, given the measures of ∠A and ∠B, find the measure of ∠C. Also, identify the type of triangle it is based on its angle measures, either acute, right, or obtuse.

7. $m\angle A = 63°$ and $m\angle B = 48°$

8. $m\angle A = 13.75°$ and $m\angle B = 76.25°$

9. $m\angle A = 36°\ 15'$ and $m\angle B = 94°\ 53'$

10. $m\angle A = 22°\ 49'\ 35''$ and $m\angle B = 41°\ 18'\ 52''$

11. Is it possible for a right triangle to have the following description? If not, why not?

a) Equilateral

b) Isosceles

c) Scalene

d) Oblique

12. Draw an isosceles right triangle and label the angle measures. →

ΔXYZ is an isosceles triangle. ∠X and ∠Y are the congruent base angles. Given one of the angle measures, find the measures of the other two.

13. $m\angle X = 48°$

14. $m\angle Y = 76°$

15. $m\angle Z = 34°$

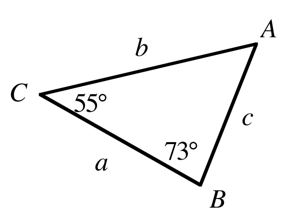
16. $m\angle Z = 105°$

17. $m\angle X = 53° 37' 49''$

18. $m\angle Z = 102° 24' 06''$

Based on the given diagram, list the sides in order from shortest to longest.

19.



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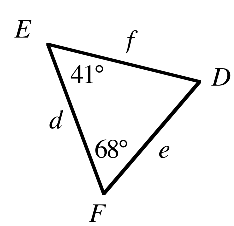
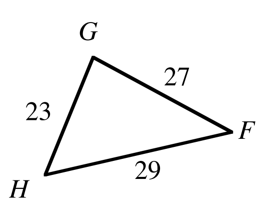


Based on the given diagram, list the angles in order from smallest to largest.

21.



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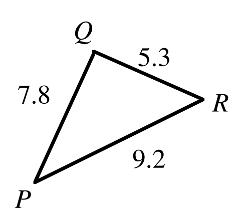

In right ΔABC, with $m\angle C = 90°$, given the lengths of two of the sides, find the length of the third side.

23. $a = 6, b = 4$

24. $a = 2, b = \sqrt{3}$

25. $a = 3, b = 3$

26. $c = 8, b = 4$

27. $c = 5\sqrt{2}, b = 5$

28. $c = 4\sqrt{3}, a = 6$

29. $a = 2\sqrt{6}, b = \sqrt{3}$

30. $a = \frac{\sqrt{3}}{2}, b = \frac{1}{2}$

Generate Pythagorean triples using the following values of m and n; Use $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$

31. $m = 2$ and $n = 1$

32. $m = 3$ and $n = 1$

33. $m = 5$ and $n = 2$