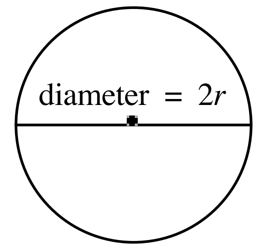
Section 1.4 Circles

THE VALUE OF π

The circumference of a circle is like the perimeter of a triangle or of a rectangle; in other words, the circumference is the measure *around* the circle.

Thousands of years ago, a question that early mathematicians struggled with was, "What is the ratio of the circumference of a circle to its diameter?" Early cultures, such as the Babylonians, Greek, and Egyptians, knew that this ratio was a constant, and as early as 2,000 BCE had estimated its value at about $3\frac{1}{8} = (3.125)$ and $4\cdot\left(\frac{8}{9}\right)^2 = (3.1605)$.



Through the ages, more and more precise measurements were taken. Some thought the ratio to be $\frac{22}{7}$, which is approximately 3.1428571. We have, of course, since learned that the actual value is an irrational number: a non-repeating, non-terminating decimal. We use the Greek letter π (pi) to represent this constant ratio.

Here is a more accurate approximation: $\pi \approx 3.1415926535...$

As you can see, the fraction $\frac{22}{7}$ is not far off for quick calculations. Also, for quick decimal calculations, π is often represented as just 3.14. For our purposes, though, we'll leave the number in its symbolic form, π.

So, the ratio of the circumference of a circle to its diameter is π:

$$\frac{\text{Circumference}}{\text{diameter}} = \pi \quad \Rightarrow \quad \frac{C}{d} = \pi \qquad \text{Multiply each side by } d.$$

$$C = \pi \cdot d \qquad \text{Use } d = 2r.$$

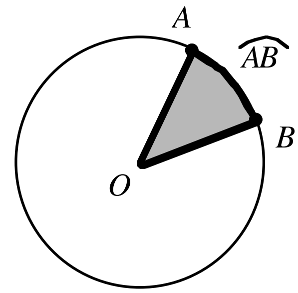
$$C = \pi \cdot (2r) \qquad \text{Simplify.}$$

$$C = 2\pi r$$

In summary,

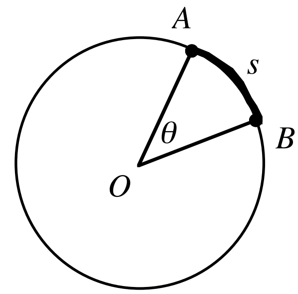
$$\boxed{\textit{Circumference} = 2\pi r}$$

The Length of an Arc



Recall from Section 1.1, an angle with its vertex at the center of a circle is called a ***central angle***. A central angle in a circle creates a ***sector*** of the circle, and it ***subtends*** an ***arc*** on the circle. For example, in the diagram, $\angle AOB$ creates arc AB and sector AOB.

We may be interested in the length of the arc and/or the area of the sector. These can be determined by setting up a proportion between the sector and the entire circle.



As shown in the second diagram at right, it is common to refer to the length of $\overset{\frown}{AB}$ as s, and the measure of the central angle as θ.

Note 1: In this section, we focus only on the length of an arc, not the area of a sector.

Note 2: θ (*theta*) is the eighth letter of the Greek alphabet.

	Arc length	Angle measure
Sector	s	θ
Whole circle	$2\pi r$	360°

$$\frac{\text{length of the arc}}{\text{circumference of circle}} = \frac{\text{measure of the central angle}}{360°}$$

$$\frac{s}{2\pi\, r} = \frac{\theta}{360°}$$

This proportion has three variables: s, r, and θ. If we know the values of two of them, such as the radius, r, and the arc length, s, then we can solve for the missing value.

Knowing the radius of a circle, we can use this proportion to find either the arc length, s, or the central angle measure, θ, depending on what other information we are given.

Caution: If not written well, s can look like 5, so it is common to use x in place of s when solving this proportion.

Example 1: A circle has a radius of 5 inches. What is the length of the arc, s, with a central angle, θ, of $90°$? (**Note:** We will use x to represent s in the equation so that s isn't mistaken for a 5.)

Procedure: $\dfrac{s}{2\pi r} = \dfrac{\theta}{360°}$ Replace r with 5 and θ with 90°, and let x represent s.

Answer: $\dfrac{x}{2 \cdot \pi \cdot 5} = \dfrac{90°}{360°}$ Simplify each side. $\dfrac{90 \text{ degrees}}{360 \text{ degrees}} = \dfrac{90}{360} = \dfrac{1}{4}$

$\dfrac{x}{10\pi} = \dfrac{1}{4}$ Isolate x by multiplying each side by $\dfrac{10\pi}{1}$.

$\dfrac{10\pi}{1} \cdot \dfrac{x}{10\pi} = \dfrac{1}{4} \cdot \dfrac{10\pi}{1}$ Simplify.

$x = \dfrac{10\pi}{4}$ Simplify.

$s = \dfrac{5\pi}{2}$ inches

You Try It 1 A circle has a radius of 9 feet. What is the length of the arc with a central angle of 120°?

Example 2: A circle has a radius of 6 cm. What is measure of the central angle, θ, that subtends an arc with length 4π cm?

Procedure: Again we use the established proportion: $r = 6$, $s = 4\pi$. The unknown value is θ.

$$\frac{s}{2\pi r} = \frac{\theta}{360°}$$ Replace r with 6 and s with 4π.

Answer:

$$\frac{4\pi}{2 \cdot \pi \cdot 6} = \frac{\theta}{360°}$$ Simplify each side.

$$\frac{4\pi}{12\pi} = \frac{\theta}{360°}$$ Simplify the left side to $\frac{1}{3}$; isolate θ by multiplying each side by $\frac{360°}{1}$.

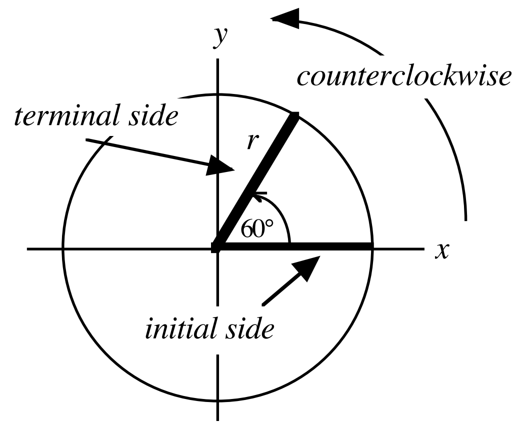
$$\frac{360°}{1} \cdot \frac{1}{3} = \frac{\theta}{360°} \cdot \frac{360°}{1}$$ Simplify.

$$\frac{360°}{3} = \theta$$ Simplify.

$$\theta = 120°$$

You Try It 2 A circle has a radius of 4 meters. What is measure of the central angle that subtends an arc with length 6π meters?

ANGLES AROUND THE CIRCLE: STANDARD POSITION

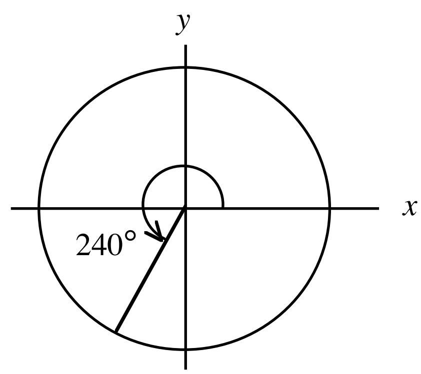


To understand more about angles around the circle we can use the Cartesian Coordinate System, the x-y-plane. We center a circle at the origin with an unspecified radius, r.

A central angle in ***standard position*** has its *initial side* on the positive x-axis. The other side of the angle is called the terminal side.

An angle is considered to be *positive* when the terminal side "sweeps" *counterclockwise* to generate the angle. The arrow representing 60° shows this counterclockwise sweep.

ANGLES MORE THAN 180° (REFLEX ANGLES)

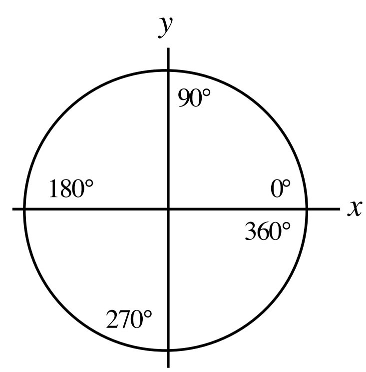


The notion that a full circle is 360° allows us to consider angle measures that are greater than 180°. For example, an angle encompassing more than half of the circle has an angle measure more than 180°.

For example, $\frac{2}{3}$ of a circle is 240°:

$$\frac{2}{3} \text{ of } 360° = \frac{2}{3} \cdot \frac{360°}{1} = \frac{2}{1} \cdot \frac{120°}{1} = 240°$$

An angle measuring between 180° and 360° is called a **reflex angle**. (It is not a commonly used term, but it's nice to know there is a word for such angles.)

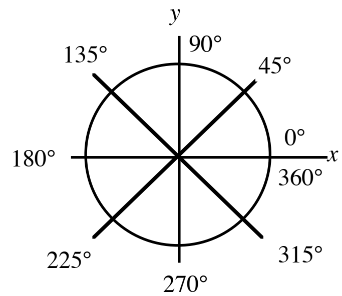


We can label special places along the circle with their corresponding angle measures.

In the diagram at right you can see the degree measures along the x-and y-axes. Notice that the circle even allows for a 0° angle.

Whereas the angle measures of a triangle are restricted to being between 0° and 180°, angle measures around a circle have no restrictions.

Recall from Section 1.1, to aid us in locating angles around the circle, we can use the horizontal and vertical radii—now referred to as the angle measures on the *x*- and *y*-axes—along with a series of 45° angles,

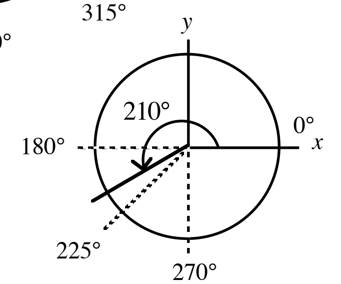


For example, to draw a 210° angle in the circle, we

i) first recognize that it is between 180° and 270°;

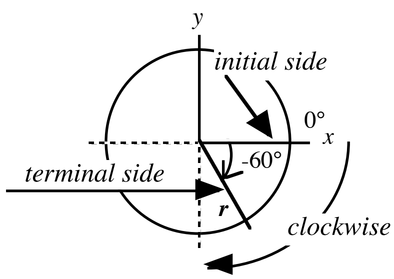
ii) refine the location by considering the half-way value, 225°.

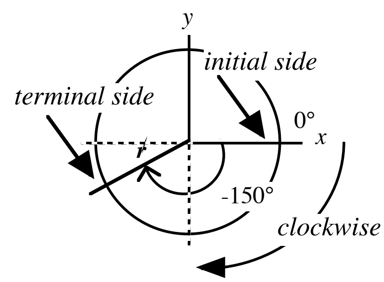
iii) 210° is between 180° and 225°, and it is closer to 225° than it is to 180°.



NEGATIVE ANGLES

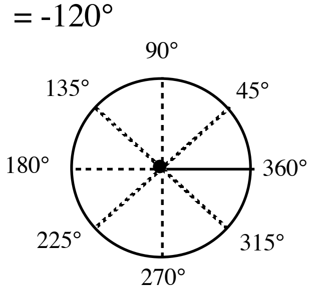
Central angles in standard position are considered *negative* when the terminal side sweeps clockwise to generate the angle. The arrow indicates this clockwise sweep.



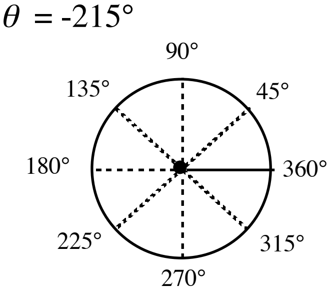


You Try It 3 For each, draw a central angle in a circle with the given number of degrees. *(Hint: Count backwards 90° at a time.)*

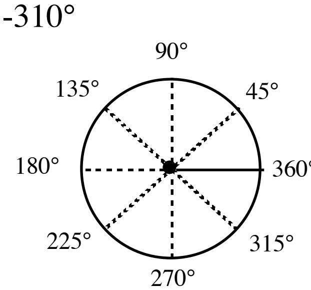
a) θ = -120°



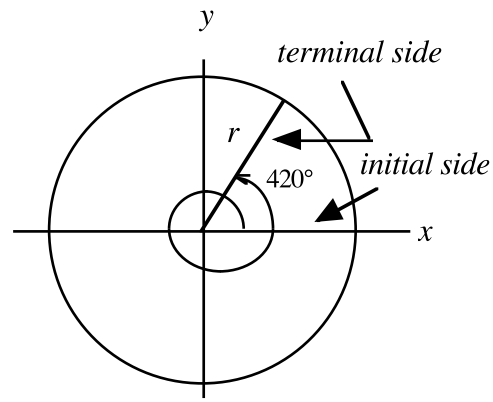
b) θ = -215°



c) θ = -310°



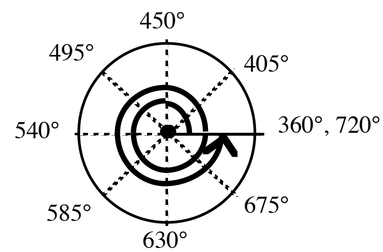
ANGLES GREATER THAN 360°



With the circle allowing us to have unusual angles—those more than 180° and negative angles—why not have angles that go around the circle more than once?

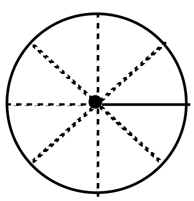
With the circle, we can consider angles that are more than one full revolution (360°), such as 420°.

At right we see the next set of angle values on the axes and at the half-way marks:

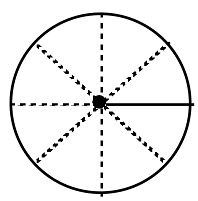


You Try It 4 For each, draw a central angle in a circle with the given number of degrees. Show the "wrapping" of the angle as it passes 360°.

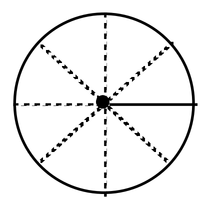
a) $\theta = 385°$



b) $\theta = 500°$

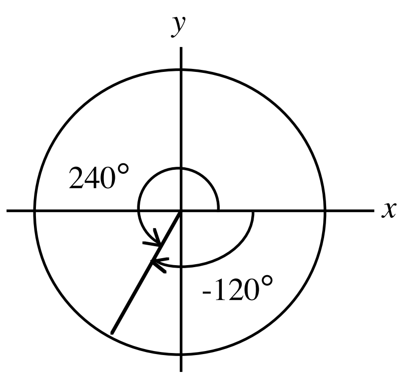


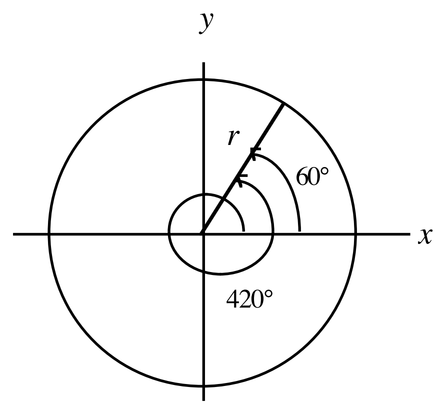
c) $\theta = 700°$



COTERMINAL ANGLES

Angles that share a common terminal side are said to be ***coterminal***. Two angles that are coterminal differ by 360° (one full circle) or a multiple of 360°, such as 720° or 1080°.





Example 3: Determine whether or not angles with these measures are coterminal.

a) 490° and 90° b) 150° and 870° c) 315° and -45°

Procedure: Find the difference between the angles. If the difference is 360° or a multiple of 360° (such as 720°), the angles are coterminal.

Answer:

a) $490° - 90° = 400°$
They are *not* coterminal; they do not differ by a multiple of 360°.

b) $870° - 150° = 720°$
They are coterminal; they differ by a multiple of 360°.

c) $315° - (-45°) = 315° + 45° = 360°$
They are coterminal; they differ by a multiple of 360°.

You Try It 5 Determine whether or not angles with these measures are coterminal.

a) 160° and 500°	b) 400° and -40°	c) -75° and -435°	d) 150° and -570°

Example 4: Find both a positive and a negative angle around the circle that is coterminal with the given angle.

a) 210° b) 600° c) -75°

Answer: Either add or subtract 360° (or a multiple of 360°) to find these coterminal angles.
(There are other answers than those shown here.)

a) $210° + 360° = \mathbf{570°}$ and $210° - 360° = \mathbf{-150°}$

570° and -150° are both coterminal with 210°

b) $600° - 360° = \mathbf{240°}$ and $240° - 360° = \mathbf{-120°}$

240° and -120° are both coterminal with 600°

c) $-75° + 360° = \mathbf{285°}$ and $-75° - 360° = \mathbf{-435°}$

285° and -435° are both coterminal with -75°

You Try It 6 Find both a positive and a negative angle around the circle that is coterminal with the given angle.

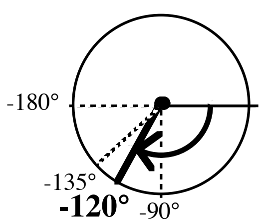
a) 160° b) 400° c) -170°

You Try It Answers

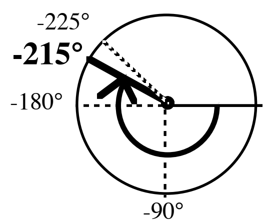
YTI 1: $s = 6\pi$ feet **YTI 2:** $\theta = 270°$

YTI 3:

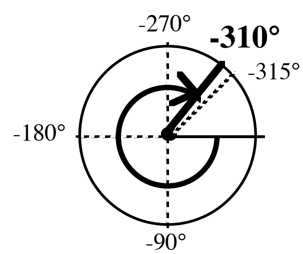
a)



b)

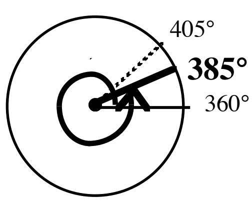


c)



YTI 4:

a)



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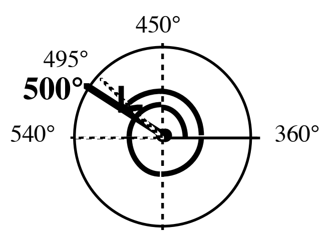
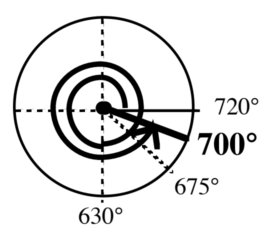


c)



YTI 5: a) **No**. 500° - 160° = 340°
b) **No**. 400° - (-40°) = 440°
c) **Yes**. -75° - (-435°) = 360°
d) **Yes**. 150° - (-570°) =720°

YTI 6: a) 520° and -200°
b) 40° and -320°
c) 190° and -530°

Section 1.4 Focus Exercises

Given the radius of a circle and the measure of a central angle, $m\angle AOB$, determine the length of the arc it subtends.

1. $r = 2,\ m\angle AOB = 180°$

2. $r = 12,\ m\angle AOB = 60°$

3. $r = 1,\ m\angle AOB = 30°$

4. $r = 8,\ m\angle AOB = 20°$

5. $r = 6,\ m\angle AOB = 40°$

6. $r = 9,\ m\angle AOB = 200°$

7. $r = \frac{1}{2},\ m\angle AOB = 150°$

8. $r = \frac{5}{2},\ m\angle AOB = 45°$

Given the radius of a circle and the measure of an arc, m Arc_{AB}, determine the measure of the central angle that subtends it.

9. $r = 2,\ m\,\text{Arc}_{AB} = \pi$

10. $r = 6,\ m\,\text{Arc}_{AB} = 2\pi$

11. $r = 3,\ m\,\text{Arc}_{AB} = 2\pi$

12. $r = 5,\ m\,\text{Arc}_{AB} = \pi$

13. $r = 1,\ m\,\text{Arc}_{AB} = \dfrac{\pi}{2}$

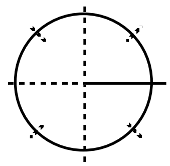
14. $r = \dfrac{1}{2},\ m\,\text{Arc}_{AB} = \dfrac{\pi}{3}$

15. $r = 8,\ m\,\text{Arc}_{AB} = 10\pi$

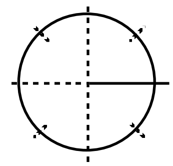
16. $r = \dfrac{5}{2},\ m\,\text{Arc}_{AB} = \dfrac{15\pi}{8}$

For each given angle measure, (i) locate it in a circle using standard position, and (ii) identify an angle ***between 0° and 360°*** *that is coterminal with it.*

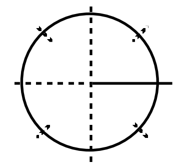
17. -60°



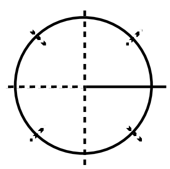
18. -200°



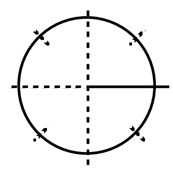
19. -270°



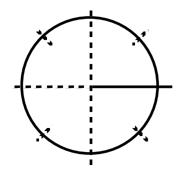
20. -90°



21. -445°

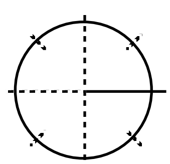


22. -800°

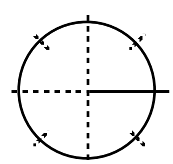


For each given angle measure, (i) locate it in a circle using standard position, and (ii) identify an angle ***between -360° and 0°*** *that is coterminal with it.*

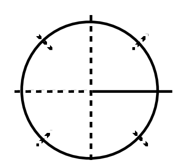
23. 100°



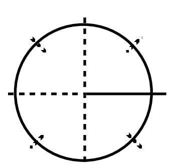
24. 240°



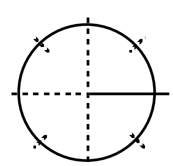
25. 180°



26. 380°



27. 490°



28. 930°