Section 7.5 Factoring Completely

Objectives
In this section, you will learn to:

- Factor a polynomial completely.

To successfully complete this section, you need to understand:

- Factoring the GCF (7.1)
- Factoring trinomials (7.3 & 7.4)
- Factoring the difference of squares (7.4)

INTRODUCTION

We have seen four techniques of factoring polynomials:

Section 7.1:	Factoring out the greatest common monomial factor (GCF)
Section 7.2:	Factoring quadrinomials by grouping
Section 7.3 & 7.4:	Factoring trinomials using the Factor Game
Section 7.4:	Factoring the difference of squares

Sometimes, two of those factoring techniques can be (*must* be) applied to factor a polynomial *completely*. This is because the polynomial might have more than just two factors, such as $x(x + 3)(x - 5)$. This is the factored form of the polynomial $x^3 - 2x^2 - 15x$.

FACTORING COMPLETELY

How do we achieve a factored form that has three factors? Often this means first identifying and extracting the GCF; then, the polynomial within the parentheses can be factored further.

Let's take a closer look at factoring $x^3 - 2x^2 - 15x$. First notice that there is an x in each of the three terms, so the GCF is x and can be extracted:

$$x^3 - 2x^2 - 15x$$

$$= x(x^2 - 2x - 15)$$

Now, the trinomial in the parentheses is possibly factorable. We won't know for sure until we play the Factor Game, so for the moment, consider only the trinomial:

$x^2 - 2x - 15$ Product = $1 \cdot (-15) = -15$, Sum = -2
Winning combination is <u>+3 and -5</u>.

Because this is a simple trinomial (the lead coefficient is 1), we can factor directly from the Factor Game: $= (x + 3)(x - 5)$

Putting these two factors with the GCF that we extracted, the full factored form of the polynomials is

$$x^3 - 2x^2 - 15x = x(x + 3)(x - 5).$$

This technique of factoring once and then factoring again, often called *factoring completely*, is used throughout this section.

Example 1 demonstrates factoring completely for a binomial.

Example 1: Factor each polynomial completely.

a) $m^3 - 25m$ b) $2w^4 + 8w^2$ c) $45x^3 - 5x$ d) $4 - y^2$

Procedure: Identify and extract the GCF, and then see if the binomial factor can, itself, be factored. Part d) should first be written in descending order.

Answer:

a)

$m^3 - 25m$	The GCF is m, so extract m from each term.
$= m(m^2 - 25)$	The binomial, $m^2 - 25$, is the difference of squares and can be factored as the product of a pair of conjugates.
$= m(m - 5)(m + 5)$	The binomial, $m^3 - 25m$, is now completely factored.

b)

$2w^4 + 8w^2$	The GCF is $2w^2$, so extract this from each term.
$= 2w^2(w^2 + 4)$	The binomial, $w^2 + 4$, is the *sum* of squares and cannot be factored further. So, the binomial, $2w^4 + 8w^2$, is now completely factored.

c)

$45x^3 - 5x$	The GCF is $5x$, so extract $5x$ from each term.
$= 5x(9x^2 - 1)$	The binomial, $9x^2 - 1$, is the difference of squares and can be factored as the product of a pair of conjugates.
$= 5x(3x - 1)(3x + 1)$	The binomial, $45x^3 - 5x$, is now completely factored.

d)

$4 - y^2$	First write this binomial in descending order.
$= -y^2 + 4$	Because the terms have no other factors in common, the GCF is 1. However, because the lead term is negative, we must extract -1 from the binomial.
$= -1(y^2 - 4)$	The binomial, $y^2 - 4$, is the difference of squares and can be factored as the product of a pair of conjugates.
$= -1(y - 2)(y + 2)$	The binomial, $4 - y^2$, is now completely factored.

You Try It 1 Factor each binomial completely. Use Example 1 as a guide.

a) $7x^3 + 28x$ b) $50y^4 - 8y^2$ c) $49 - p^2$

Here are some examples of factoring trinomials completely.

Example 2: Factor each trinomial.

a) $3x^4 + 15x^3 - 18x^2$ b) $4v^2 - 6v + 10$ c) $4p + 30 - 2p^2$

Procedure: First, check for a common monomial that can be extracted.
Second, see if the polynomial within the parentheses can be factored.
Third, make sure that all factors are shown in the final factored form.

Answer: a) $3x^4 + 15x^3 - 18x^2$

The GCF for this trinomial is $3x^2$.
Extract $3x^2$ from each term.

$= 3x^2(x^2 + 5x - 6)$

Use the Factor game to see if the trinomial can factor.

Product = $1 \cdot (-6) = -6$ and Sum = $+5$.

The winning combination is -1 and +6.

Because we got a winning combination for the Factor Game, this *simple* trinomial does factor We can put the numbers of the winning combination directly into the binomial factors, $(x - 1)(x + 6)$.

The final factored form is

$= 3x^2(x - 1)(x + 6)$

The polynomial is completely factored.

b) $4v^2 - 6v + 10$ — The GCF for this trinomial is 2. Extract 2 from each term.

$= 2(2v^2 - 3v + 5)$ — Use the Factor game to see if the trinomial can factor.

Product = $2 \cdot 5 = +10$ and Sum = -3.

There is no winning combination.

The trinomial, $2v^2 - 3v + 5$, cannot be factored, but we still have a factored form of the original polynomial:

$= 2(2v^2 - 3v + 5)$ — The polynomial is completely factored.

c) $3x + 28 - x^2$ — The GCF for this trinomial is $3x^2$. Extract $3x^2$ from each term.

$= 3x^2(x^2 + 5x - 6)$ — Use the Factor game to see if the trinomial can factor.

Product = $1 \cdot (-6) = -6$ and Sum = + 5.

The winning combination is -1 and +6.

Because we got a winning combination for the Factor Game, this *simple* trinomial does factor We can put the numbers of the winning combination directly into the binomial factors, $(x - 1)(x + 6)$.

The final factored form is

$= 3x^2(x - 1)(x + 6)$ — The polynomial is completely factored.

d) $4p + 30 - 2p^2$ — First write this trinomial in descending order.

$= -2p^2 + 4p + 30$ — The terms have a GCF of 2, but because the lead term is negative, we must extract -2 from the trinomial.

$= -2(p^2 - 2p - 15)$ — This is a simple trinomial, and the winning combination of the Factor Game is -5 and +3, so we can put these numbers into the binomial factors directly.

$= -2(p - 5)(p + 3)$ — The trinomial is now completely factored.

Consider the following trinomial: $4x^2 + 20x + 24$. What if we did not notice the common factor of 4 and we don't extract it? Is the trinomial still factorable?

Yes, but the work involved to factor it will be more challenging. Consider the trinomial as it is written. This is not a simple trinomial, so we can play the Factor Game and use factor by grouping, just as we did in Section 7.3:

$4x^2 + 20x + 24$	Product = $4 \cdot (24)$ = +96 and Sum = +20 Winning Combination is <u>+8 and +12</u>.	
$= 4x^2 + 8x + 12x + 24$		96
$= (4x^2 + 8x) + (12x + 24)$	Factor each grouping.	/ \ 1 96 2 48 3 32
$= 4x(x + 2) + 12(x + 2)$	The common binomial factor is $(x + 2)$.	4 24 6 16 8 12 ✓
$= (4x + 12)(x + 2)$		

The factoring is correct, but it is not complete because both terms in the first binomial, $4x$ and 12, have a common factor of 4 that can be extracted. This makes the final factored form

$= 4(x + 3)(x + 2)$

However, if we had noticed the common factor of 4 at the beginning, the whole factoring process would have been a lot simpler, as demonstrated here:

$4x^2 + 20x + 24$	Extract the GCF, 4.
$= 4(x^2 + 5x + 6)$	Use the Factor game to see if the trinomial can factor:
	Product = $1 \cdot (6)$ = +6 and Sum = +5 Winning Combination is <u>+3 and +2</u>.

Because the trinomial is a simple trinomial, we can place the results of the Factor Game directly into the binomial factors:

$= 4(x + 3)(x + 2)$

The point is this, whether we extract the GCF of a trinomial as the first step or as the last step, we get the same final factored form. However, the product and sum numbers of the Factor Game are usually much easier to work with if we extract the GCF as the first step, rather than as the last.

You Try It 2 Factor each binomial completely. Use Example 2 as a guide.

a) $5x^3 - 30x^2 + 45x$

b) $15 - 3y^2 + 12y$

Section 7.5 Answers to the You Try It Exercises

You Try It 1: a) $7x(x^2 + 4)$ b) $2y^2(5y - 2)(5y + 2)$

c) $-1(p - 7)(p + 7)$

You Try It 2: a) $5x(x - 3)^2$ b) $-3(y - 5)(y + 1)$

Section 7.5 Focus Exercises

Factor completely.

1. $y^4 - 9y^2$

2. $8c^3 - 50c$

3. $5x^3 + 20x$

4. $10m^4 - 40m^2$

5. $12x^3 - 27x$

6. $2y^3 + 72y$

7. $36 - p^2$

8. $25 - y^2$

9. $16v - v^3$

10. $100r - r^3$

11. $2x - 32x^3$

12. $75p^2 - 27p^4$

13. $x^3 - 13x^2 - 30x$

14. $x^3 - 7x^2 - 8x$

15. $5x^2 - 15x + 20$

16. $4x^2 - 32x + 60$

17. $10x^4 + 20x^3 - 80x^2$

18. $3x^5 + 24x^4 + 36x^3$

19. $-2x^3 + 22x^2 - 36x$

20. $-6x^2 + 24x - 18$

21. $-5x^3 - 10x^2 + 15x$

22. $-9w^3 - 18w^2 - 9w$

23. $x^2 - 20x + x^3$

24. $-12p^2 + p^4 + p^3$

25. $5x + 6 - x^2$

26. $24 - 5y - y^2$

27. $9x^2 - 30x + 24$

28. $12y^3 - 48y^2 + 45y$

29. $-6x^3 + 14x^2 + 12x$

30. $-10x^2 + 14x + 12$

Think Outside the Box.

Factor completely.

31. $x^4 - 16$

32. $y^4 - 81$

33. $16p^4 - 81$

34. $1 - w^4$

35. $6w^4 - 18w^3 + 15w^2 - 45w$

36. $18mp^3 - 12mp^2 - 30mp + 20m$