Section 5.4 Factoring Trinomials, $a > 1$

INTRODUCTION

Another important feature of algebra is factoring trinomials. Some examples of trinomials that can be factored are:

a) $2x^2 + 7x + 6$ b) $3x^2 - 20x + 12$

c) $6x^2 + 7x - 5$ d) $x^2 - 5x - 24$

(Each of these trinomials is of the form $\mathbf{ax^2 + bx + c}$.
We'll be referring to this form throughout this section.)

If you look closely at each of those trinomials, you'll notice that none of them have a common monomial factor that we can factor out, as those found in Section 5.2. And yet, each of those trinomials can be written as a product of two binomials.

Here is each one written in a factored form:

a) $2x^2 + 7x + 6 = (x + 2)(2x + 3)$

b) $3x^2 - 20x + 12 = (3x - 2)(x - 6)$

c) $6x^2 + 7x - 5 = (2x - 1)(3x + 5)$

d) $x^2 - 5x - 24 = (x + 3)(x - 8)$

How do we know that those trinomials factor that way? How can we test to see if what is shown is the correct factoring?

We simply need to multiply the binomials together, using the FOIL method, to see that the factoring is accurate. Look below; each binomial product is shown multiplied out. The only thing left to do is to combine the like terms. (Each middle term—on the left side—is underlined just as the two like terms—on the right side—are underlined.)

	Trinomial	Factored Form		The Product	The FOIL products
a)	$2x^2 \underline{+ 7x} + 6 =$	$(x + 2)(2x + 3)$	because	$(x + 2)(2x + 3) =$	$2x^2 \underline{+ 3x + 4x} + 6$
b)	$3x^2 \underline{- 20x} + 12 =$	$(3x - 2)(x - 6)$	because	$(3x - 2)(x - 6) =$	$3x^2 \underline{- 18x - 2x} + 12$
c)	$6x^2 \underline{+ 7x} - 5 =$	$(2x - 1)(3x + 5)$	because	$(2x - 1)(3x + 5) =$	$6x^2 \underline{+ 10x - 3x} - 5$
d)	$x^2 \underline{- 5x} - 24 =$	$(x + 3)(x - 8)$	because	$(x + 3)(x - 8) =$	$x^2 \underline{- 8x + 3x} - 24$

By the way, some trinomials cannot be factored. If they cannot be factored, then they are considered to be **prime**. Some examples of prime trinomials are:

(a) $x^2 + 5x + 30$ and (b) $x^2 - 15x + 24$

These will be explored later.

WHAT ARE FACTORS?

Let's review what you should know about factoring.

FACTORS IMPLY MULTIPLICATION

Whenever we refer to factors, we are—either directly or indirectly—referring to a *product*, a multiplication.

To consider factors of 15, for example, we think of the numbers that *multiply* to get 15, namely 3 and 5 (or 1 and 15). We're <u>not</u> looking for numbers that *add* to get 15, like 7 and 8.

Yet, doesn't it seem strange when we multiply $(x + 2)(x + 3)$ and get $x^2 + 5x + 6$? Where's the multiplication in $x^2 + 5x + 6$? Why is there so much addition in $x^2 + 5x + 6$? Isn't that supposed to be a product, a multiplication?

Remember that multiplication is an abbreviation for repeated addition; in other words, multiplication is based on addition, the two really are inseparable. Even when multiplying numbers, like 12 x 13, addition plays a role. Look. Here is the product of 12 and 13 written in both a familiar way and an unusual way. Notice that addition plays a role in each of them.

The Familiar Way		An Unusual Way
12	→	10 + 2
x 13	→	x 10 + 3
36	→	30 + 6
+ 120	→	+ 100 + 20
156	→	100 + 50 + 6

Notice that the product 156 can be written out as $100 + 50 + 6$, a sum (addition). Yet, it's still the product of 12 and 13 (meaning 12 and 13 are factors of 156: $156 = 12 \cdot 13$).

Notice this, too. If we treat $12 \cdot 13$ in a more algebraic way, we could write it as $(10 + 2)(10 + 3)$, and its product could be written as $10^2 + 5\cdot10 + 6$. This is just like $(x + 2)(x + 3) = x^2 + 5x + 6$ (where the x has been replaced by 10).

The description presented above is intended to show you that many trinomials can be factored into the product of two binomials. We simply need to learn how to do it.

Factoring Trinomials and "FOIL"

The understanding behind factoring trinomials is the FOIL method for multiplying two binomials. Take, for example, $(3x + 2)(2x + 5)$. In multiplying this out, using the FOIL method, we get the four **FOIL products**:

$$\begin{array}{cccc} \textbf{F} & \textbf{O} & \textbf{I} & \textbf{L} \\ & & & \\ 6x^2 & \underline{+\ 15x} & \underline{+\ 4x} & +\ 10 \end{array}$$

and it's the middle two terms (the **O**UTER and **I**NNER products) that combine to give the middle term of the trinomial:

$$= \ 6x^2 \ \underline{+\ 19x} \ + \ 10$$

The process for factoring the trinomial is really a game of numbers. Look closely at both the middle term of the trinomial, **19x**, and the two underlined terms in the FOIL products, **15x** and **4x**:

$$15x + 4x = 19x \qquad \text{and} \qquad 15x \cdot 4x = 60x^2$$

Actually, it's the *numbers* that matter most:

$$15 + 4 = \mathbf{19} \qquad \text{and} \qquad 15 \cdot 4 = \mathbf{60}$$

It's probably more obvious how the **19** is related, but what is the significance of the **60**?

Thinking numbers, 60 happens to be the product of the first and last terms of the trinomial (the product of the numbers in **F** and **L** in the four FOIL products), $\mathbf{6 \cdot 10} = 60$.

Notice that *60* is found as a **product**, $15 \cdot 4 = 60$, and *19* is found as a **sum**, $15 + 4 = 19$. These two numbers, *60* and *19*, turn out to be the **Key #** and the **Sum #** of **The Factor Game**, first introduced in Section 1.5 and later revisited in Section 5.3.

Let's see how this works.

Identifying the Key and Sum Numbers of the Factor Game

In a short while, we'll be applying the Factor Game to factoring trinomials of the form

$$ax^2 + bx + c.$$

In order for the Factor Game to be useful, **a** must be a positive integer, while **b** and **c** could be either positive or negative integers. In other words, the trinomial might look like $5x^2 - 4x - 12$. In this case, we would interpret the coefficients and constant to be

$$a = 5, \qquad b = -4 \qquad \text{and} \qquad c = -12.$$

We will be using these values (a, b and c) to develop the Key and Sum numbers for the Factor Game. It's rather straight forward:

$$\boxed{\text{the Key \# } = a \cdot c \ \text{ and the Sum \# } = b}$$

Example 1: Given the trinomial, identify the values of the coefficients and the constant (a, b and c) and use them to identify the Key and Sum numbers. Also, find the solution for the Key and Sum # combination.

a) $3x^2 - 14x + 8$

$a = 3$
$b = -14$
$c = 8$

Key # = $3\cdot(8) = 24$
Sum # = -14

The solution is **- 12** and **- 2**

b) $5x^2 - 4x - 12$

$a = 5$
$b = -4$
$c = -12$

Key # = $5\cdot(-12) = -60$
Sum # = -4

The solution is **6** and **- 10**

You probably don't need to write the values of a, b and c separately; it's okay to do that part in your head. Also, it's possible that we put the Key and Sum numbers together correctly, but there is no solution.

Example 2: Given the trinomial, identify the Key and Sum numbers. Also, find the solution for the Key and Sum # combination, if possible.

a) $10x^2 + 11x + 3$

Key # = $10\cdot(3) = 30$
Sum # = 11

The solution is **5** and **6**

b) $6x^2 + 5x - 6$

Key # = $6\cdot(-6) = -36$
Sum # = 5

The solution is **9** and **- 4**

c) $3x^2 + 6x - 8$

Key # = $3\cdot(-8) = -24$
Sum # = 6

There is **no solution**

Exercise 1: Given the trinomial, identify the Key and Sum numbers. Also, find the solution for the Key and Sum # combination.

a) $4x^2 + 12x + 5$

Key # = Sum # =

Solution is ______________

b) $5x^2 - 13x + 6$

Key # = Sum # =

Solution is ______________

c) $10x^2 - 3x - 4$

Key # = Sum # =

Solution is ______________

d) $3x^2 + 4x - 4$

Key # = Sum # =

Solution is ______________

e) $2x^2 + x - 15$

Key # = Sum # =

Solution is ______________

f) $6x^2 - x - 2$

Key # = Sum # =

Solution is ______________

g) $x^2 - 5x + 4$

Key # = Sum # =

Solution is ______________

h) $x^2 - x - 20$

Key # = Sum # =

Solution is ______________

i) $x^2 + 8x + 16$

Key # = Sum # =

Solution is ______________

j) $9x^2 + 12x + 4$

Key # = Sum # =

Solution is ______________

PUTTING THE <u>FACTOR GAME</u> TO GOOD USE: FACTOR BY GROUPING

The whole point of the Factor Game is to be able to rewrite a trinomial into four terms. When we find the solution to the factor game, what we have found is a way to split the trinomial's middle term into two terms, thereby making it a four-term polynomial.

Consider the trinomial $8x^2 + 10x - 3$. Identify the **Key** and **Sum** numbers and find the answer to the Factor Game.

(1) The set up: $8x^2 + 10x - 3$ **Error!**

The solution is + 12 and - 2. This means that the middle term, + 10x, can be rewritten as $\underline{+12x - 2x}$. In doing so, the trinomial is rewritten with four terms:

$8x^2 + 10x - 3$

(2) Write the trinomial with four terms: $= 8x^2 + 12x - 2x - 3$

As a four-term polynomial, we can factor it by using Factor by Grouping:

(3) Use parentheses to get two groupings: $= (8x^2 + 12x) + (-2x - 3)$

(4) Factor each individual grouping: $= 4x\mathbf{(2x + 3)} + -1\mathbf{(2x + 3)}$

(5) Factor out (2x + 3) from each: $= (4x - 1)\mathbf{(2x + 3)}$

So, the factors of $8x^2 + 10x - 3$ are $\mathbf{(4x - 1)}$ and $\mathbf{(2x + 3)}$.

Example 3: Factor each trinomial by using the Factor Game to rewrite it as a four-term polynomial. Then use factor by grouping.

$10x^2 + 11x + 3$ **Error!**

Here, again, is the original trinomial: $10x^2 + 11x + 3$

Split the middle term into $+5x + 6x$: $= 10x^2 + 5x + 6x + 3$

Now show the groupings: $= (10x^2 + 5x) + (6x + 3)$

Now factor out the common monomial factor from each group: $= 5x(2x + 1) + 3(2x + 1)$

Now factor out $(2x + 1)$: $= \mathbf{(5x + 3)(2x + 1)}$

Of course, this can also be written in the other order: $= \mathbf{(2x + 1)(5x + 3)}$

We can now say that, <u>the factors of $10x^2 + 11x + 3$ are $(5x + 3)$ and $(2x + 1)$.</u>

Let's see some other examples. There may be more work here (shown below) than you might do, or you might choose to organize your work a little differently. In any case, please make sure you are clear in your work so that others can follow what you've done.

Example 4: Factor each trinomial by using the Factor Game to rewrite it as a four-term polynomial. Then use factor by grouping.

a) $6x^2 + 5x - 6$ Key # = $6 \cdot (-6) = -36$, Sum # = $+5$. The solution is $+9$ and -4

The original trinomial:		$6x^2 + 5x - 6$
Split the middle term into $+9x - 4x$:	$=$	$6x^2 + 9x - 4x - 6$
Now show the groupings:	$=$	$(6x^2 + 9x) + (-4x - 6)$
Now factor out the common monomial factor from each group:	$=$	$3x(2x + 3) + -2(2x + 3)$
Now factor out $(2x + 3)$:	$=$	$\mathbf{(3x - 2)(2x + 3)}$

We can now say that, the factors of $6x^2 + 5x - 6$ are $(3x - 2)$ and $(2x + 3)$.

b) $5x^2 - 16x + 12$ Key # = $5 \cdot 12 = 60$ Sum # = -16. The solution is -10 and -6

The original trinomial:		$5x^2 - 16x + 12$
Split the middle term into $-10x - 6x$:	$=$	$5x^2 - 10x - 6x + 12$
Now show the groupings:	$=$	$(5x^2 - 10x) + (-6x + 12)$
Now factor out the common monomial factor from each group:	$=$	$5x(x - 2) + -6(x - 2)$
Now factor out $(x - 2)$:	$=$	$\mathbf{(5x - 6)(x - 2)}$

We can now say that, the factors of $5x^2 - 16x + 12$ are $(5x - 6)$ and $(x - 2)$.

c) $3x^2 + 6x - 8$ $\left.\begin{array}{l}\text{Key \# } = 3 \cdot (-8) = -24 \\ \text{Sum \# } = +6\end{array}\right\}$ There is **no solution**

Since there is no solution to the Factor Game, the trinomial is ***prime*** and cannot be factored.

Exercise 2 Factor each trinomial by using the Factor Game to rewrite it as a four-term polynomial. Then use factor by grouping.

a) $4x^2 + 12x + 5$

b) $5x^2 - 13x + 6$

c) $10x^2 - 3x - 4$

d) $3x^2 + 4x - 4$

e) $2x^2 + x - 15$

f) $6x^2 - x - 2$

g) $x^2 - 5x + 4$

h) $x^2 + 8x + 16$

Answers to each Exercise

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Exercise 1:

a) Key # = + 20, Sum # = + 12; solution is + 10 and + 2
b) Key # = + 30, Sum # = - 13; solution is - 10 and - 3
c) Key # = - 40, Sum # = - 3; solution is + 5 and - 8
d) Key # = - 12, Sum # = + 4; solution is + 6 and - 2
e) Key # = - 30, Sum # = + 1; solution is + 6 and - 5
f) Key # = - 12, Sum # = - 1; solution is + 3 and - 4
g) Key # = + 4, Sum # = - 5; solution is - 4 and - 1
h) Key # = - 20, Sum # = - 1; solution is - 5 and + 4
i) Key # = + 16, Sum # = + 8; solution is + 4 and + 4
j) Key # = + 36, Sum # = + 12; solution is + 6 and + 6

Exercise 2:

a) $(2x + 1)(2x + 5)$
b) $(5x - 3)(x - 2)$
c) $(2x + 1)(5x - 4)$
d) $(3x - 2)(x + 2)$
e) $(2x - 5)(x + 3)$
f) $(2x + 1)(3x - 2)$
g) $(x - 4)(x - 1)$
h) $(x + 4)(x + 4)$

Section 5.4 Focus Exercises

Factor each trinomial by using the Factor Game to rewrite it as a four-term polynomial. Then use factor by grouping.

1. $4x^2 + 11x + 6$

2. $2x^2 - 9x + 9$

3. $10x^2 + 9x + 2$

4. $x^2 - 12x + 36$

5. $4x^2 + 16x + 7$

6. $6x^2 - 17x + 5$

7. $5x^2 - 18x + 9$

8. $12x^2 - 16x + 5$

9. $6x^2 + 7x - 5$

10. $8x^2 + 10x - 3$

11. $10x^2 + 3x - 4$

12. $15x^2 + 4x - 3$

13. $12x^2 - 4x - 5$

14. $6x^2 + 7x - 10$

15. $x^2 - 9x - 36$

16. $x^2 - 4x - 32$

17. $x^2 + 11x + 36$

18. $x^2 - 12x + 27$

19. $4x^2 + 12x + 9$

20. $3x^2 - 11x - 20$