Section 0.3 The Order of Operations

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OPERATIONS

Let's be reminded of those operations seen thus far in the course:

Operation	Example	Special name
Multiplication	$3 \cdot 5 = 15$	this is a **product**
Division	$14 \div 2 = 7$	this is a **quotient**
Subtraction	$12 - 9 = 3$	this is a **difference**
Addition	$2 + 8 = 10$	this is a **sum**
Exponent	$2^3 = 8$	this is a **power**
Radical	$\sqrt{25} = 5$	this is a **square root**

The words *product*, *quotient*, *difference* and *sum* each has two meanings in mathematics; *for example*,

a **product** is both the expression (written form) *and* the result of applying multiplication:

the **product** of 5 and 3 is *written* **5 · 3**; applying multiplication, the **product** of 5 and 3 is **15.**

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EVALUATING A NUMERICAL EXPRESSION

To **evaluate** a numerical expression means to "find the value of" the expression. Consider this expression: $3 + 4 \cdot 5$. How should it be evaluated?

(1) If we apply *addition first*, and then apply multiplication, we get

$3 + 4 \cdot 5$ Should we add first?

$= 7 \cdot 5$

$= 35$

(2) If we apply *multiplication first* and then apply addition, we get

$= 3 + 4 \cdot 5$ Should we multiply first?

$= 3 + 20$

$= 23$

In other words, based upon which operation is applied first, we get two different results. However, math is an exact science and doesn't allow for two different values of the same numerical expression. So we must, instead, develop a system—a set of guidelines—that will lead each of us to the same result every single time it is applied correctly. This system is called the **Order of Operations**.

The Order of Operations is one of the most important rules you will learn in mathematics. It gives definite guidelines as to which operation should be applied first, which one second, etc. In fact, whenever a formula is used—in forestry, nursing, police work, statistics (the list goes on and on)—the order of operations must also be used.

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GROUPING SYMBOLS

The most famous pair of grouping symbols is the pair of parentheses. Parentheses create a **quantity**, suggesting that what's inside belongs to itself. The parentheses also act as a barrier, a protector of sorts, to outside influence until the value within is known.

Remember (from Section 0.1) that any expression within grouping symbols must be applied first, before any other operations are applied.

There are a variety of symbols that we can use to group two or more values. The most common is the parentheses ( ), but we also find use for brackets [ ] and (rarely) braces { }.

The radical—the square root—is also considered a grouping symbol: $\sqrt{\ }$; it groups any quantity that is within it. For example, $\sqrt{5 + 11}$ could be written as $\sqrt{(5 + 11)}$. (The parentheses are not needed but are there for emphasis.)

Also, there are basically two types of grouping symbols:

- those that form a quantity, like ( ) and [ ], and
- those that are an actual operation, like $\sqrt{\ }$

So, what *is* the order that the operations should be applied? Perfect timing; look:

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THE ORDER OF OPERATIONS

The Order Of Operations	
1.	Evaluate within all grouping symbols (one at a time), if there are any.
2.	Apply any exponents.
3.	Apply multiplication and division *reading from left to right*..
4.	Apply addition and subtraction *reading from left to right*..

We sometimes refer to the order of the operations by their **rank**. For example, we might say that the exponent has a "higher rank" than multiplication.

Similarly, multiplication and division have the same rank; that is why we must apply them (carefully) from *left to right*. You'll see how this works in some of the examples.

Exercise 1: Is it ever possible to give addition a higher rank than multiplication? Explain.

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Does The Order Of Operations Have To Be That Way?

Let's take a look back at the original expression presented at the beginning of this section, $3 + 4 \cdot 5$. We tried two different approaches:

(1) If we apply *addition first*, and then apply multiplication, we get

$$\begin{aligned} & 3 + 4 \cdot 5 \quad \text{Should we add first?} \\ = \ & 7 \cdot 5 \\ = \ & 35 \end{aligned}$$

(2) If we apply *multiplication first* and then apply addition, we get

$$\begin{aligned} = \ & 3 + 4 \cdot 5 \quad \text{Should we multiply first?} \\ = \ & 3 + 20 \\ = \ & 23 \end{aligned}$$

We now know that the second approach is the accurate one because multiplication has a higher rank than addition, so we must apply multiplication first.

Why is it that way? Just because someone said so? What if we took away the order of operations and were allowed to apply only addition and nothing else. Could it be done?

Let's take a look at what the expression $3 + 4 \cdot 5$ really means. Since multiplication is an abbreviation for repeated addition, we could think of it as:

$$\begin{aligned} & 3 \ + \ \text{four 5's} \\ = \ & 3 \ + \ (5 + 5 + 5 + 5) && \text{Let's expand the abbreviation.} \\ = \ & 3 \ + \ 5 \ + \ 5 \ + \ 5 \ + \ 5 && \text{The associative property of addition says we don't need parentheses if the only operation is addition.} \\ = \ & 23 \end{aligned}$$

Because multiplication is an abbreviation for addition, we needed to expand it first to get only addition. Another way would be to recognize that four 5's is 20 without expanding it. In other words, we can apply multiplication (it's a lot faster) directly without having to expand it first. All in all, this just tells us why multiplication has to come before addition; it is why multiplication has a higher rank than addition.

Applying the Order of Operations

The best way to understand these rules (guidelines) is to put them to work. Basically, we'll find that there is only one way to evaluate an expression using the rules, but we'll also find that some steps can be combined in certain situations. For now, though, let's evaluate each expression **one step at a time**.

Important note: When looking through Example 1, be sure to notice how the steps are presented in the answer. Notice that, in each step, only one operation is applied. Please try to imitate this process so that your work is done in the same way. Also, if an operation has *not yet* been applied, then it should be shown on the next line.

Example 1: Evaluate each according to the *Order of Operations*.

a) $3 + 4 \cdot 5$ b) $(3 + 4) \cdot 5$ c) $7 + 3^2$

d) $(7 + 3)^2$ e) $24 \div 4 \cdot 2$ f) $24 \div (4 \cdot 2)$

Procedure: Each of these has two operations; some of them have grouping symbols that will affect the order that the operations are applied.

a) $3 + 4 \cdot 5$ Two operations: addition and multiplication; multiply first

$= 3 + 20$ Notice that the addition sign appears in the second step; that's because it hasn't been applied yet.

$= 23$

b) $(3 + 4) \cdot 5$ Here are the same two operations as above, this time with grouping symbols.

$= (7) \cdot 5$ Since we've already evaluated within the grouping symbols, we really don't need the parentheses any more, but they're okay.

$= \boxed{35}$

c) $7 + 3^2$ Two operations: addition and exponent; apply the exponent first, then add.

$= 7 + 9$

$= 16$

d) $(7 + 3)^2$ Same two operations as in (c); work within the grouping symbols first.

$= (10)^2$

$= \boxed{100}$

e) $24 \div 4 \cdot 2$ two operations: division and multiplication; since they have the same "rank," and there are no grouping symbols to tell us which to apply first, we need to apply them in order from *left to right*.

$= 6 \cdot 2$

That means that <u>division</u> gets applied first

$= \boxed{12}$

f) $24 \div (4 \cdot 2)$ This time we do have grouping symbols, so there is no guess about which operation gets to be applied first.

$= 24 \div 8$ we don't need the parentheses around the 8 because there's nothing to group

$= \boxed{3}$

Exercise 2: Evaluate each according to the Order of Operations. First identify the two operations, then identify which is to be applied first. Be neat in your work and show all steps.

a) $24 \div 6 + 2$

b) $24 \div (6 + 2)$

c) $10 - 3 \cdot 2$

d) $(10 - 3) \cdot 2$

e) $12 \div 2^2$

f) $(12 \div 2)^2$

Some expressions have more than two operations. In those situations, we need to be even more careful when we apply the order of operations.

Example 2: Evaluate each according to the *Order of Operations*.

a) $36 \div 3 \cdot 6 - 2$ b) $36 \div (3 \cdot 6) - 2$ c) $36 \div [3 \cdot (6 - 2)]$

Procedure: Each of these has three operations; when we have a quantity within a larger grouping, we'll usually use the brackets, [ ], as the "larger" grouping symbols. As the expression gets evaluated, it isn't always necessary to keep them as brackets.

a) $36 \div 3 \cdot 6 - 2$ Though temptation might tell you otherwise, you must first divide.

$= 12 \cdot 6 - 2$

$= 72 - 2$ Notice that we are applying only one operation at a time and rewriting everything else. Patience is a virtue.

$= 70$

b) $36 \div (3 \cdot 6) - 2$ Apply the multiplication first

$= 36 \div 18 - 2$ now apply division

$= 2 - 2$ now subtract

$= 0$

c) $36 \div [3 \cdot (6 - 2)]$ We must first work inside the large brackets; inside those grouping symbols is a smaller quantity, so we must work inside of it first.

$= 36 \div (3 \cdot 4)$

$= 36 \div 12$ At the second step, $3 \cdot 4$ must still be grouped until it is evaluated.

$= 3$

Exercise 3: Evaluate each according to the *Order of Operations*. First think about the order in which the operations should be applied.

a) $35 \div 5 + 2 \cdot 3$

b) $35 \div (5 + 2) \cdot 3$

c) $(6 + 12) \div (2 \cdot 3)$

d) $6 \div [12 \div (2 + 4)]$

e) $(2 \cdot 3)^2 \div (6 + 3)$

f) $2 \cdot 3^2 \div (6 + 3)$

Sometimes an expression will have two sets of grouping symbols that are "unrelated" to each other, meaning that evaluating one does not affect the evaluation of the other. In other words, some quantities can be evaluated at the same time.

For example, in the expression $(8 - 3) \cdot (12 \div 4)$ we can evaluate within each grouping symbol regardless of what operation each one has inside:

	$(8 - 3) \cdot (12 \div 4)$	Here there are three operations: subtraction, multiplication and division.
$=$	$(5) \cdot (3)$	Subtraction and division have EQUAL rank here because of the parentheses.
$=$	15	

Now let's look at some examples that contain the radical. Remember, the radical—the square root—is both a grouping symbol and an operation. As an operation, it gets applied before exponents (and multiplication, etc.) because it is also a grouping symbol.

Example 3: Evaluate each completely. Remember, the radical is both a grouping symbol and an operation.

a) $\sqrt{5 + 11}$ b) $\sqrt{3^2 + 4^2}$ c) $13 - 2 \cdot \sqrt{9}$

Procedure: The radical and the absolute value are both grouping symbols and operations.

a) $\sqrt{5 + 11}$ First apply the addition.

$= \sqrt{16}$ Now apply the radical (square root).

$= 4$

b) $\sqrt{3^2 + 4^2}$ Apply both exponents within the same step.

$= \sqrt{9 + 16}$ Next apply the addition.

$= \sqrt{25}$ Lastly, apply the radical (square root).

$= 5$

c) $13 - 2 \cdot \sqrt{9}$ The radical is the only grouping symbol, but there is nothing

$= 13 - 2 \cdot 3$ to evaluate *inside*; however, since the radical *is* a grouping

$= 13 - 6$ symbol, and an operation, we apply it—the square root—first.

$= 7$

Exercise 4: Evaluate each according to the *Order of Operations*. Again, think about the order in which the operations should be applied.

a) $3 + \sqrt{16}$

b) $9 \cdot \sqrt{25}$

c) $\sqrt{100 - 36}$

d) $\sqrt{4 \cdot 9}$

e) $\sqrt{1 + 12 \cdot 4}$

f) $\sqrt{(6 - 2) \cdot 5^2}$

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THE DISTRIBUTIVE PROPERTY

It's now time to introduce a property that brings both multiplication and addition into the expression at the same time. It is called the **Distributive Property**. It is called this because, as you will see, we actually distribute one number to two (or more) numbers, just as a mail carrier might distribute the same advertisement to two neighbors' houses.

Believe it or not, you have actually used the distributive property quite extensively; maybe more so when you were young, but you may have come across it relatively recently. Remember how you use to (and maybe still do) multiply, say, $6 \cdot 17$? Let's look:

6	
x 17	(17 is, of course, 7 + 10)
4 2	Multiply the 6 times the 7; put zero under the two and
+ 6 0	multiply the 6 times the 1 (it's really times *10*: $6 \cdot 10 = 60$)
1 0 2	Add to get the final result.

In algebra, however, that process might look more like this next example. Look at the similarities between this one is to the previous one. (same numbers, different methods)

$6 \cdot 17$	
$= 6 \cdot (7 + 10)$	Rewrite 17 as (7 + 10)
$= 6 \cdot 7 + 6 \cdot 10$	distribute the "6 · " through to both numbers;
$= 42 + 60$	multiply the 6 times the 7; multiply the 6 times the 10, the same as above.
$= 102$	Add to get the final result

The question that you might have right now is, "Is THIS how we have to do multiplication, now?" (whine, whine, whine…) No, it isn't. This process is simply being used to illustrate the distributive property,

$$b \cdot (c + d) = b \cdot c + b \cdot d$$

or $$b(c + d) = bc + bd.$$

However, the distributive property is not restricted to addition only; it can just as easily be used for subtraction as well:

$$b(c - d) = bc - bd.$$

We can also distribute "from the right" if we need to:

$(c + d)b = cb + db.$ This can also be written as:

$= bc + bd.$ The commutative property allows us to switch the order of the multiplication.

In summary,

The Distributive Property		
1.	**b(c + d)**	**= bc + bd**
2.	**(c + d)b**	**= cb + db**

Notice that what is actually being distributed is more than just a number, **b**; the multiplication symbol is being distributed along with the b; so really, "**b ·**" (*b times*) is being distributed. We also have a name for the number that is distributed; it is called the **multiplier**.

The number that is distributed is called the **multiplier**.

Let's practice using the distributive property:

Example 4: Use the distributive property to evaluate each expression.

a) $5(7 + 3)$ b) $2(4 + 6)$ c) $(5 - 2)8$

Answer:

a) $5(7 + 3)$ — 5 is the *multiplier*, and, as there is nothing between it and the quantity, the operation is automatically multiplication.

$= 5\cdot(7 + 3)$ — This step isn't necessary, but it shows the multiplication.

$= 5\cdot7 + 5\cdot3$ — Distribute the "5 times" through to both terms.

$= 35 + 15$ — Multiply and then add.

$= 50$

b) $2(4 + 6)$ — The *multiplier* is 2.

$= 2\cdot4 + 2\cdot6$ — Actually, even this step isn't necessary; it's just showing what needs to be done, but this can be done in your head.

$= 8 + 12$

$= 20$

c) $(5 - 2)8$ — This time, only the necessary steps are included.

$= 40 - 16$ — You can distribute and multiply directly without writing the multiplication step down.

$= 24$

As accurate as the distributive property is, it still leads to some confusion. We've been learning about the order of operations in this section, and the first thing we learn is "Apply the operation inside parentheses first."

When an expression is all numerical the distributive property isn't necessary; in fact, it's rarely used. The distributive property is more important to algebra. Later in the course you'll be re-introduced to the distributive property as it relates to algebra.

For the purposes of practicing the distributive now, when it's relatively easy, do this next exercise as directed, using the distributive property.

Exercise 5: Use the distributive property to evaluate these. **SHOW ALL STEPS.** (If you wish, you may *check* your answers using the order of operations.)

a) $5\cdot(6 + 3)$ b) $2\cdot(8 - 1)$

c) $(10 + 4)\cdot 6$ d) $(11 - 5)\cdot 4$

Why the Distributive Property Works

Let's see why the distributive property works the way it does. Consider the example $\mathbf{3} \cdot (2 + 6)$.

Since this is multiplication, and multiplication is an abbreviation for repeated addition, this means

"The sum of **three** (2 + 6)'s."

This can be written as $= (2 + 6) + (2 + 6) + (2 + 6)$.

The associative property can be used to write that expression without parentheses:

$$= 2 + 6 + 2 + 6 + 2 + 6$$

We can then use the commutative property to reorganize the numbers:

$$= 2 + 2 + 2 + 6 + 6 + 6$$

Let's group the 2's separately from the 6's: $= (2 + 2 + 2) + (6 + 6 + 6)$

Each grouping has *three* of something: = **three** 2's + **three** 6's

$$= \mathbf{3} \cdot 2 + \mathbf{3} \cdot 6$$

Hence, from the beginning: $\mathbf{3} \cdot (2 + 6) = \mathbf{3} \cdot 2 + \mathbf{3} \cdot 6$.

Behold, the distributive property.

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Answers to each Exercise

Section 0.3

Exercise 1: Yes, addition can have a higher rank than multiplication if the addition is with grouping symbols.

Exercise 2:

a)	6	b)	3	c)	4
d)	14	e)	3	f)	36

Exercise 3:

a)	13	b)	15	c)	3
d)	3	e)	4	f)	2

Exercise 4:

a)	7	b)	45	c)	8
d)	6	e)	7	f)	10

Exercise 5:

a) $5\cdot(6 + 3)$
$= 5\cdot6 + 5\cdot3$
$= 30 + 15$
$= 45$

b) $2\cdot(8 - 1)$
$= 2\cdot8 - 2\cdot1$
$= 16 - 2$
$= 14$

c) $(10 + 4)\cdot6$
$= 10\cdot6 + 4\cdot6$
$= 60 + 24$
$= 84$

d) $(11 - 5)\cdot4$
$= 11\cdot4 - 5\cdot4$
$= 44 - 20$
$= 24$

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Section 0.3 Focus Exercises

1. Evaluate each according to the *Order of Operations*. Simplify just one step at a time, one operation at a time. Show all work.

a) $30 \div 5 + 1$

b) $8 + 5 \cdot 2$

c) $5 \cdot 3^2$

d) $6^2 - \sqrt{25}$

e) $\left[12 + 28\right] \div (7 - 3)$

f) $\left[(6 - 2) \cdot 3\right]^2$

2. Evaluate each according to the *Order of Operations*. Simplify just one step at a time, one operation at a time. Show all work.

a) $2^3 \cdot 3^2$

b) $3 \cdot \sqrt{9 + 7}$

c) $\sqrt{6^2 - (5 \cdot 4)}$

d) $\sqrt{8 \cdot 6 + 1}$

e) $(\sqrt{6 - 5} + 4)^2$

f) $(\sqrt{32 + 4})^2$

3. Use the distributive property to evaluate these. **SHOW ALL STEPS.**

a) $8 \cdot (9 - 2)$

b) $5 \cdot (11 + 3)$

c) $(6 + 5) \cdot 7$

d) $(12 - 1) \cdot 3$

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