Section 1.1 Definitions and Properties

Objectives
In this section, you will learn to:

- Abbreviate repeated addition using multiplication.
- Abbreviate repeated multiplication using an exponent.
- Identify factors pairs of a number.
- Translate the basic operations to and from English.
- Apply the Order of Operations.
- Identify and apply the Commutative, Associative, and Distributive Properties.
- Identify and apply the identities for addition and multiplication.

To successfully complete this section, you need to understand:

- Exponents and Square Roots (R.1)

INTRODUCTION

Abbreviations are shortcuts that allow us to quickly communicate words or ideas. When sending text messages, for example, we might use LOL for "laugh out loud," or BRB for "be right back."

Algebra is all about using and manipulating symbols and abbreviations. For example, the English expression *the sum of ten and six* can be abbreviated as the numerical expression $\underline{10 + 6}$. We also use letters, called *variables*, to represent numbers. For example, *the sum of a number and six* can be abbreviated as the algebraic expression $\underline{n + 6}$, where n represents an unspecified number.

The Greek mathematician Diophantus was one of the first to use symbols to *abbreviate* mathematical expressions and thought. Though his symbols were different from what we use today, he is credited with taking the discussion of mathematics and problem solving from sentence form into symbolic form. Could you imagine having to *discuss* everyday mathematics without the use of a plus sign or a times sign?

Letting symbols represent mathematical ideas allows us to abbreviate those ideas. A mathematical sentence written in words (English, not Greek) can be more easily understood if it is written symbolically, as long as we know what the symbols mean.

TWO IMPORTANT ABBREVIATIONS

There are two important abbreviations that we use throughout the study of algebra: *multiplication* and the *exponent*.

1. Multiplication is an abbreviation for repeated addition. For example, $5 \cdot 3$ means *the sum of five 3's*: $3 + 3 + 3 + 3 + 3$.

$$5 \cdot 3 \text{ is the abbreviated form and } 3 + 3 + 3 + 3 + 3 \text{ is the expanded form.}$$

2. Likewise, the exponent is an abbreviation for repeated multiplication. For example, 2^4 means *four factors of 2*: $2 \cdot 2 \cdot 2 \cdot 2$.

2^4 is the abbreviated form and $2 \cdot 2 \cdot 2 \cdot 2$ is the expanded form.
The abbreviated form, 2^4, is also called *exponential form*.

Example 1: State the meaning of each abbreviation and write it in its expanded form.

a) $3 \cdot 8$ b) $4 \cdot x$ c) 9^4 d) y^5

Answer:

a) The sum of three 8's: $\underline{8 + 8 + 8}$ b) The sum of four x's: $\underline{x + x + x + x}$

c) Four factors of 9: $\underline{9 \cdot 9 \cdot 9 \cdot 9}$ d) Five factors of y: $\underline{y \cdot y \cdot y \cdot y \cdot y}$

<u>You Try It 1</u> State the meaning of each abbreviation and write it in its expanded form. Use Example 1 as a guide.

a) $5 \cdot 7$ b) $6 \cdot p$ c) 2^7 d) m^3

BASIC TERMINOLOGY

Algebra is a generalized form of arithmetic in which we often use letters to represent numbers. These letters, such as x and y, are called **variables**. In algebra, as in arithmetic, there are four basic **operations**: addition (+), subtraction (–), multiplication (x and ·), and division (÷).

We also have the operations of power (base$^{\text{exponent}}$) and the square root (radical), $\sqrt{}$. The *written form* of an operation, such as **5 + 4**, is called an **expression**.

When an expression is contained within grouping symbols, such as parentheses, the expression is called a **quantity**. A quantity represents a single value.

When an expression contains only numbers and operations, such as $6 \cdot 4 - 3$, it is called a *numerical expression*. If an expression contains variables, such as $6x - 3$, it is called an *algebraic expression*.

There are also words for the <u>results</u> of the operations, as shown in the chart below. To get the result of an operation we must *apply* the operation. For example, in the expression **5 + 4**, we apply addition and get a result of 9.

Operation	**Name**	**As an expression**	**Result**	**Meaning:**
Addition **(plus)**	**Sum**	$3 + 5$	$= 8$	The sum of 3 and 5 is 8.
Subtraction **(minus)**	**Difference**	$9 - 5$	$= 4$	The difference of 9 and 5 is 4.
Multiplication **(times)**	**Product**	$2 \cdot 3$	$= 6$	The product of 2 and 3 is 6.
Division **(divided by)**	**Quotient**	$6 \div 3$	$= 2$	The quotient of 6 and 3 is 2.
Exponent	**Power**	2^3	$= 8$	The 3rd power of 2 is 8.
Radical	**Square root**	$\sqrt{16}$	$= 4$	The square root of 16 is 4.

Note: In algebra, multiplication, such as 3 x 5, is often represented by a raised dot, $3 \cdot 5$. We also use parentheses to represent multiplication, such as 3(5) or (3)5, and occasionally (3)(5). When using parentheses, the raised dot is not required.

Note: A sum is both the *written expression* of addition and the *result* of applying addition:

- The sum of 3 and 5 is written $3 + 5$ (expression), and
- the sum of 3 and 5 is 8 (result).

This dual definition of an operation is also true for subtraction, multiplication, and division.

Example 2: Write each as a mathematical expression and find the result.

a) The difference of 8 and 3. b) The quotient of 35 and 5. c) The second power of 10.

Answer: a) $8 - 3 = 5$ b) $35 \div 5 = 7$ c) $10^2 = 100$

You Try It 2 Write each as a mathematical expression and find the result. Use Example 2 as a guide.

a) The product of 5 and 8. b) The sum of 9 and 3. c) The square root of 9.

We can use a variable to represent an unspecified number. For example, *the sum of 5 and a number can be represented* by $5 + x$.

Example 3: Write each as a mathematical expression.

a) The product of 12 and a number.
b) The difference of a number and 9.
c) The square of a number.

Procedure: You may use any letter to represent *a number*. For each exercise shown here, a different variable is used.

Answer:

a) $12 \cdot y$
b) $w - 9$
c) p^2

You Try It 3 Write each as a mathematical expression. Use Example 3 as a guide.

a) The quotient of a number and 3.
b) The fourth power of a number.
c) The sum of a number and 10.

We can use the names of the operations to write numerical and algebraic expressions in English. For example, the expression $8 + 2$ can be written as *the sum of 8 and 2*. If an expression contains a variable, write *a number* when writing the expression in English. For example, the expression $x + 2$ can be written as *the sum of a number and 2*.

Note: The four basic operations all require the word *and* in their translation, but *power* and *square root* do not.

Example 4: Write each expression in English.

a) $15 - 6$
b) $24 \div y$
c) $p \cdot 12$
d) 3^8

Procedure: The word *and* must be used in the translation for parts a), b), and c).

Answer:

a) The difference of 15 and 6.
b) The quotient of 24 and a number.
c) The product of a number and 12.
d) The eighth power of 3.

You Try It 4 Write each expression in English. Use Example 4 as a guide.

a) $18 \div 6$
b) $25 + w$
c) $3 \cdot 16$
d) $\sqrt{y}$

FACTORS

Any whole number can be written as the product of two factors, which is called a **factor pair**.

For example,
- 12 can be written as $2 \cdot 6$, so $2 \cdot 6$ is a factor pair of 12.
- 9 can be written as $3 \cdot 3$, so $3 \cdot 3$ is a factor pair of 9.
- 7 can be written as $1 \cdot 7$, so $1 \cdot 7$ is a factor pair of 7.

Many numbers have more than one factor pair, and we can use a *factor pair table* to help us find all of the factors of a particular number.

Consider this factor pair table of 24. Notice how it is organized to find all of the factor pairs of 24.

24	
1	24
2	12
3	8
4	6

Start with 1 on the left and write the other factor of the pair, 24, on the right. Do the same for 2, 3, and so on. Do not include 5, because 5 isn't a factor of 24.

Now we can list all of the factors of 24: 1 and 24; 2 and 12; 3 and 8; 4 and 6.

Example 5: Use a factor pair table to find all of the factor pairs of 30.

Procedure: Think of 30 as a product of two numbers. Start with 1 on the left side, then 2, then 3, and so on, and decide whether those numbers are factors of 30.

Answer:

30	
1	30
2	15
3	10
5	6

$1 \cdot 30 = 30$
$2 \cdot 15 = 30$
$3 \cdot 10 = 30$
$5 \cdot 6 = 30$

Notice that 4 is not on this list because 4 doesn't divide evenly into 30.

The factor pairs of 30 are 1 and 30; 2 and 15; 3 and 10; 5 and 6.

You Try It 5 Use a factor pair table to find all of the factor pairs of each number. Use Example 5 as a guide.

a) 12

b) 16

c) 18

d) 20

THE ORDER OF OPERATIONS

To **evaluate** a mathematical expression means to find the value of the expression. When an expression contains more than one operation, there may be some confusion as to which operation is to be applied first.

Consider, for example, the expression $3 + 4 \cdot 5$. How could it be evaluated?

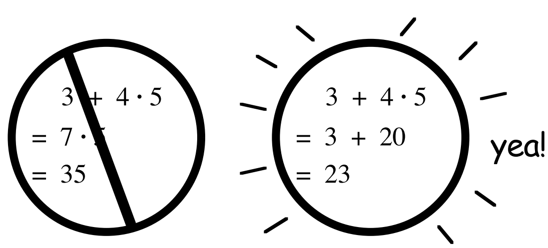
If we apply the operation of addition first, and then multiplication, we get

$$\begin{aligned} & 3 + 4 \cdot 5 \\ =\ & 7 \cdot 5 \\ =\ & \mathbf{35} \quad \textit{(Is this correct?)} \end{aligned}$$

If, however, we evaluate using multiplication first and then addition, we get

$$\begin{aligned} & 3 + 4 \cdot 5 \\ =\ & 3 + 20 \\ =\ & \mathbf{23} \quad \textit{(Or is this correct?)} \end{aligned}$$

Notice that, depending on which operation is applied first, we get two different results. Math, however, is an exact science and doesn't allow for two different values of the same expression. As we will learn, the only correct way to evaluate $3 + 4 \cdot 5$ is to multiply first, so the value 23 is correct.



Let's find out why.

To determine which operation should be applied first, a set of guidelines, called the **Order of Operations**, was established. The order of operations was developed with these thoughts in mind:

1. Any quantity within grouping symbols should be evaluated first. Grouping symbols includes parentheses ( ), brackets [ ], braces { }, the radical (the square root symbol) $\sqrt{\ }$, and the fraction bar (division bar) $\frac{\text{numerator}}{\text{denominator}}$.

2. Because an exponent is an abbreviation for repeated multiplication, exponents should rank higher than multiplication.

3. Because multiplication and division are inverse operations, they should be ranked together. Also, because multiplication is an abbreviation for repeated addition, multiplication (and division) should rank higher than addition.

4. Because addition and subtraction are inverse operations, they should be ranked together.

In summary,

The Order Of Operations

1. Evaluate within all grouping symbols, if there are any.
2. Apply any exponents.
3. Apply multiplication and division *from left to right*.
4. Apply addition and subtraction *from left to right*.

Note: There are basically two types of grouping symbols:

- Those that form a quantity, like ( ), [ ], and { }, and
- those that are actual operations, like $\sqrt{\ }$ and the fraction bar $\frac{\text{numerator}}{\text{denominator}}$.

The radical is *both* a grouping symbol *and* an operation. The same is true for the fraction bar.

We sometimes refer to the order of the operations by their *rank*. For example, we might say that an exponent has a higher rank than multiplication.

Because multiplication and division have the same rank, we must apply them (carefully) from *left to right*. The same is true for addition and subtraction. These rules will be demonstrated in the examples that follow.

Think About It 1: When evaluating an expression, is it ever possible to apply addition before multiplication? Explain your answer.

____

____

The best way to understand the order of operation guidelines is to put them to work. We'll find that there is only one way to evaluate an expression using the rules, but we'll also find that some steps can be combined in certain situations. For now, though, let's evaluate each expression one step at a time.

Note:
As shown in Example 5, the equal sign in front of each new step indicates that one step is equivalent to the next. In this case, the equal signs are called *continued equal signs*.

Continued equal signs indicate equivalent expressions, and do not indicate equations, which are presented in Chapter 2.

Example 6: Evaluate each expression.

a) $14 - 6 \div 2$ b) $(14 - 6) \div 2$ c) $7 + 3^2$

d) $(7 + 3)^2$ e) $24 \div 4 \cdot 2$ f) $24 \div (4 \cdot 2)$

Procedure: Use the order of operations applying one operation at a time. First, identify the two operations and determine which is to be applied first.

Answer: a) $14 - 6 \div 2$ Two operations, subtraction and division: divide first.

$= 14 - 3$ Notice that the minus sign appears in the second step. That's because it has not yet been applied.

$= \boxed{11}$

b) $(14 - 6) \div 2$ Here are the same two operations as above, this time with grouping symbols. Evaluate the expression inside the parentheses first.

$= 8 \div 2$ Because we've already evaluated within the grouping symbols, we don't need the parentheses any more.

$= \boxed{4}$

c) $7 + 3^2$ Two operations, addition and exponent: apply the exponent first, then add.

$= 7 + 9$

$= \boxed{16}$

d) $(7+3)^2$ — The same two operations as in (c); work within the grouping symbols first.

$= (10)^2$ — $(10)^2$ is an abbreviation for $10 \cdot 10$, which is 100.

$= \boxed{100}$

e) $24 \div 4 \cdot 2$ — Two operations: division and multiplication. Because they have the same rank, and there are no grouping symbols, we must apply them in order from *left to right*. This means that division is applied first.

$= 6 \cdot 2$

$= \boxed{12}$

f) $24 \div (4 \cdot 2)$ — This time we have grouping symbols, so we begin by evaluating the expression within the parentheses.

$= 24 \div 8$

$= \boxed{3}$

You Try It 6 Evaluate each according to the order of operations. Show all steps! (First identify the two operations, then identify which is to be applied first.) Use Example 6 as a guide.

a) $24 \div 6 + 2$

b) $24 \div (6 + 2)$

c) $10 - 3 \cdot 2$

d) $(10 - 3) \cdot 2$

e) $12 \div 2^2$

f) $(12 \div 2)^2$

Some expressions contain more than two operations. In those situations, we must be more careful—and show more work—when we apply the order of operations.

Example 7: Evaluate each according to the order of operations.

a) $36 \div 3 \cdot 6 - 2$

b) $36 \div (3 \cdot 6) - 2$

c) $36 \div [3 \cdot (6 - 2)]$

Answer: Each of these has three operations. In part c) there is a smaller quantity $(6 - 2)$ within the larger quantity of the brackets, [ ].

a) $36 \div 3 \cdot 6 - 2$ — Because multiplication and division have the same rank, we apply them from left to right. We divide first.

$= 12 \cdot 6 - 2$ — Notice that we are applying only one operation at a time and rewriting everything else

$= 72 - 2$

$= 70$

b) $36 \div (3 \cdot 6) - 2$ — Evaluate the expression within the grouping symbols first.

$= 36 \div 18 - 2$ — Divide.

$= 2 - 2$ — Subtract.

$= 0$

c) $36 \div [3 \cdot (6 - 2)]$ — Start with what is inside the large brackets. Inside those grouping symbols is another quantity, and we must evaluate it first: $6 - 2 = 4$.

$= 36 \div [3 \cdot 4]$ — Evaluate within the brackets: $3 \cdot 4 = 12$.

$= 36 \div 12$ — Divide.

$= 3$

Example 7(c) illustrates that when one quantity is within another one, the inner quantity is to be evaluated first.

You Try It 7 Evaluate each according to the order of operations. Identify the order in which the operations should be applied. On each line, write the operation that should be applied and then apply it. Use Example 7 as a guide.

a) $36 \div 3 + 3 \cdot 2$

b) $36 \div (3 + 3) \cdot 2$

c) $36 \div (3 + 3 \cdot 2)$

d) $11 + 4 \cdot 6 - 1$

e) $11 + [4 \cdot (6 - 1)]$

f) $2 \cdot 3^2 \div (6 + 3)$

As stated earlier, a radical is both a grouping symbol and an operation. This means that we must first evaluate within the radical, if appropriate. For example, in the expression $\sqrt{25 - 9}$, we must first subtract to get $\sqrt{16}$. Then we can apply the square root and get 4.

However, if an expression contains a radical with no internal operation, such as $36 \div \sqrt{16} + 5$, we must first evaluate $\sqrt{16}$ because the radical is a grouping symbol and has the highest rank.

Example 8: Evaluate each completely.

a) $\sqrt{5 + 11}$ b) $\sqrt{3^2 + 4^2}$ c) $13 - 2 \cdot \sqrt{9}$

Procedure: The radical is both a grouping symbol and an operation. In parts a) and b), we must evaluate within the radical first. In part c), we apply the radical before applying any other operation.

Answer:

a) $\sqrt{5 + 11}$ First apply addition.

$= \sqrt{16}$ Now apply the square root.

$= 4$

b) $\sqrt{3^2 + 4^2}$ Apply both exponents within the same step.

$= \sqrt{9 + 16}$ Next apply addition.

$= \sqrt{25}$ Apply the square root.

$= 5$

c) $13 - 2 \cdot \sqrt{9}$ The radical is a grouping symbol, so we must apply it—

$= 13 - 2 \cdot 3$ the square root—first. Next, apply multiplication and then

$= 13 - 6$ subtraction.

$= 7$

You Try It 8 Evaluate each according to the order of operations. Use Example 8 as a guide.

a) $\sqrt{4 \cdot 9}$ b) $\sqrt{25} - \sqrt{9}$ c) $\sqrt{1 + (12 \cdot 4)}$

d) $\sqrt{(6 - 2) \cdot 5^2}$

DOUBLE QUANTITIES

Sometimes an expression contains two sets of grouping symbols that are unrelated to each other. So, evaluating within one set does not affect the evaluation within the other. This means that some quantities can be evaluated at the same time.

For example, in the expression $(8 - 3) \cdot (12 \div 4)$ we can evaluate within each grouping symbol regardless of what operation each contains:

$(8 - 3) \cdot (12 \div 4)$ There are three operations: subtraction, multiplication, and division. Because the parentheses have the highest rank, we can apply subtraction and division separately, yet at the same time.

$= (5) \cdot (3)$

$= 15$

Example 9: Evaluate each according to the *order of operations*.

a) $(5 \cdot 6) \div (4 - 7)$ b) $(-8 + 3) \cdot (-14 \div 2)$ c) $(24 \div 6) - \sqrt{5 + 11}$

Procedure: Single operations within different grouping symbols can be applied at the same time.

Answer:

a) $(5 \cdot 6) \div (4 - 7)$ We can apply both the multiplication and the subtraction in the same step.

$= (30) \div (-3)$ A positive divided by a negative results in a negative.

$= -10$

b) $(-8 + 3) \cdot (-14 \div 2)$ We can apply both the addition and the division in the same step.

$= (-5) \cdot (-7)$ A negative multiplied by a negative results in a positive.

$= +35 = 35$

c) $(24 \div 6) - \sqrt{5 + 11}$ We can apply both the division and the addition in the same step.

$= 4 - \sqrt{16}$ Now apply the radical.

$= 4 - 4$ Subtract.

$= 0$

You Try It 9 Evaluate each expression according to the order of operations. Use Example 9 as a guide.

a) $(2 \cdot 3) + (42 \div 6)$ b) $(13 - 6) \cdot (5 + 2)$ c) $(6 + 3)^2 - \sqrt{5 - 4}$

THE DIVISION BAR

As stated earlier, the fraction bar groups the numerator (top) separately from the denominator (bottom). When evaluating an expression that involves a fraction bar, we treat the numerator and denominator as if they were grouped separately and evaluate within them separately.

$$\frac{(\text{numerator})}{(\text{denominator})}$$

For example, in the expression $\frac{3 \cdot 8}{5 + 1}$, the operations multiplication (in the numerator) and addition (in the denominator) can be applied at the same time because of the grouping provided by the division bar:

$$\frac{(3 \cdot 8)}{(5 + 1)}.$$

This expression becomes $\frac{24}{6}$, and we can apply division, 24 ÷ 6, to get 4.

> **Note:**
>
> The parentheses shown in $\frac{(3 \cdot 8)}{(5 + 1)}$ are not necessary; they are there to emphasize the grouping nature of the division bar.

Example 10: Evaluate each.

a) $\frac{13 - 1}{28 \div 7}$

b) $\frac{\sqrt{16} + 5}{7 - 2^2}$

c) $30 \div \frac{\sqrt{36}}{8 - 5}$

Procedure: The numerator and the denominator are separate quantities and must be evaluated separately.

Answer: a) $\frac{13 - 1}{28 \div 7}$ Apply subtraction in the numerator and division in the denominator.

$= \frac{12}{4}$ Think of this as division: $12 \div 4 = 3$.

$= 3$

b) $\frac{\sqrt{16} + 5}{7 - 2^2}$ First apply the radical in the numerator and square the 2 in the denominator.

$= \frac{4 + 5}{7 - 4}$ Next apply subtraction and addition.

$= \frac{9}{3}$ Think of this as division: $9 \div 3 = 3$.

$= 3$

c)	$30 \div \dfrac{\sqrt{36}}{8\ -\ 5}$	Simplify the fraction completely before applying the first division. Apply the radical in the numerator and subtract in the denominator.
	$= \ 30 \div \dfrac{6}{3}$	Next simplify the fraction: $6 \div 3 = 2$.
	$= \ 30 \div 2$	Divide.
	$= \ 15$	

You Try It 10 Evaluate each expression according to the order of operations. Use Example 10 as a guide.

a) $\dfrac{2 \cdot 3}{42 \div 7}$ b) $\dfrac{4+14}{3^2}$ c) $\dfrac{\sqrt{9}\ +\ 5}{4\ -\ 2}$ d) $\dfrac{\sqrt{100}\ +\ 2^2}{3^2 -\ 2}$

THE COMMUTATIVE PROPERTIES

If you have a \$5 bill and a \$10 bill in your wallet, the order in which they are placed in your wallet does not matter because they still combine to equal \$15: 5 + 10 = 15 and 10 + 5 = 15. These two sums illustrate a simple, yet important, property of mathematics, the *Commutative Property of Addition*.

The **Commutative Property of Addition** states: When adding two numbers, it does not matter which number is written first; the resulting sum will be the same.

The Commutative Property is true for multiplication as well. For example, $3 \cdot 4 = 12$ and $4 \cdot 3 = 12$.

Here are the Commutative Properties in their non-numerical form. We use letters, *a* and *b*, to represent any two numbers:

The Commutative Properties of Addition and Multiplication

If a and b are any numbers, then

Addition	Multiplication
$a + b = b + a$	$a \cdot b = b \cdot a$
The order in which we add two numbers doesn't affect the resulting sum.	The order in which we multiply two numbers doesn't affect the resulting product.

Caution: Division is *not* commutative. If we switch the order of the numbers when dividing, then we don't get the same result. For example, $10 \div 5$ is *not* the same as $5 \div 10$. Here's an illustration:

At a youth car wash, if 5 youths wash a truck for \$10, they earn \$2 each (\$10.00 ÷ 5 = \$2.00); however, if 10 youths wash a car for \$5, they earn only \$0.50 each (\$5.00 ÷ 10 = \$0.50).

Similarly, subtraction is *not* commutative.

Example 11: If possible, use a commutative property to rewrite each expression.

a) $3 + 4$ b) $6 \cdot 5$ c) $9 - 5$ d) $15 \div 3$

Procedure: Switch the order of the numbers to arrive at an equivalent expression.

Answer:

a) $3 + 4 = \underline{4 + 3}$ b) $6 \cdot 5 = \underline{5 \cdot 6}$

c) Subtraction is not commutative.

d) Division is not commutative.

You Try It 11 If possible, use a commutative property to rewrite each expression. Use Example 11 as a guide.

a) $2 + 6$ b) $9 - 2$ c) $12 \div 3$ d) $4 \cdot 5$

THE ASSOCIATIVE PROPERTIES

Another important property of mathematics is the *Associative Property*. There is an Associative Property for addition and one for multiplication.

Whenever we need to add three numbers together, such as 3 + 2 + 4, we must always choose to add two of them first. We can use parentheses to group two of them to create a quantity, then evaluate the quantity first.

For example, with the sum $3 + 2 + 4$, we can either group the first two numbers or the last two numbers:

a) $(3 + 2) + 4$ or b) $3 + (2 + 4)$

Add 3 and 2 first. Add 2 and 4 first.

$= 5 + 4$ $= 3 + 6$

$= 9$ $= 9$

Notice that the resulting sum, 9, is the same no matter which grouping we choose.

The same is true of multiplying three numbers together, such as $3 \cdot 2 \cdot 4$:

c) $(3 \cdot 2) \cdot 4$ or d) $3 \cdot (2 \cdot 4)$

Multiply 3 times 2 first. Multiply 2 times 4 first.

$= 6 \cdot 4$ $= 3 \cdot 8$

$= 24$ $= 24$

Notice that the resulting product, 24, is the same no matter which grouping we choose.

The Associative Properties of Addition and Multiplication

If $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are any three numbers, then

Addition	**Multiplication**
$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
If the only operation is addition, we can change the grouping of the numbers without affecting the resulting sum.	If the only operation is multiplication, we can change the grouping of the numbers without affecting the resulting product.

Example 12: Write each expression two different ways using parentheses, then evaluate it.

a) $1 + 6 + 3$ b) $5 \cdot 2 \cdot 3$

Procedure: Use the Associative Property; notice that the order in which the numbers are written doesn't change.

Answer:

a) $\underline{1 + 6 + 3} = \underline{\overset{7 \quad + 3}{(1 + 6) + 3}} = \underline{\overset{1 + \quad 9}{1 + (6 + 3)}} = \underline{\mathbf{10}}$

b) $\underline{5 \cdot 2 \cdot 3} = \underline{\overset{10 \quad \cdot 3}{(5 \cdot 2) \cdot 3}} = \underline{\overset{5 \cdot \quad 6}{5 \cdot (2 \cdot 3)}} = \underline{\mathbf{30}}$

You Try It 12 Write each expression two different ways using parentheses, then evaluate it. Use Example 12 as a guide.

a) $\underline{2 + 8 + 6}$ = ________________ = ______________ = __________

b) $\underline{6 \cdot 5 \cdot 2}$ = ________________ = ______________ = __________

The Associative Properties will be used throughout this text in a variety of areas. Sometimes we'll combine the Associative and the Commutative Properties to move numbers around within an expression.

For example, to multiply $5 \cdot 17 \cdot 2$, it is simplest to multiply 5 and 2 before multiplying 17:

$5 \cdot 17 \cdot 2$	The Commutative Property allows us to switch the order of $17 \cdot 2$.
$= 5 \cdot 2 \cdot 17$	The Associative Property allows us to group the first two numbers.
$= (5 \cdot 2) \cdot 17$	Multiply within the parentheses.
$= 10 \cdot 17$	Complete the multiplication.
$= 170$	

Caution: The Associative Property does *not* hold true for division and subtraction. Consider these examples, and evaluate each expression using the order of operations.

Division

a) $(24 \div 6) \div 2$
$= 4 \div 2$
$= 2$

b) $24 \div (6 \div 2)$
$= 24 \div 3$
$= 8$

$(24 \div 6) \div 2 \neq 24 \div (6 \div 2)$

Subtraction

c) $(10 - 4) - 3$
$= 6 - 3$
$= 3$

d) $10 - (4 - 3)$
$= 10 - 1$
$= 9$

$(10 - 4) - 3 \neq 10 - (4 - 3)$

These demonstrate that the Associative Property cannot be applied to either division or subtraction.

IDENTITY NUMBERS

The notion of *identity* is another important property of addition and multiplication. An **identity** is a number that, when applied, won't change the value of another number or quantity.

For addition, the identity is 0 (zero), because

$6 + \mathbf{0} = 6$ and $\mathbf{0} + 6 = 6$ 0 is called the **additive identity**.

For multiplication, the identity is 1 (one), because

$5 \cdot \mathbf{1} = 5$ and $\mathbf{1} \cdot 5 = 5$ 1 is called the **multiplicative identity**.

The Identity Numbers for Addition and Multiplication

If ***a*** is any number, then

Addition

$a + 0 = 0 + a = a$

Adding 0 to a number doesn't change the value of the expression.

Multiplication

$a \cdot 1 = 1 \cdot a = a$

Multiplying a number by 1 doesn't change the value of the expression.

Notice that each identity is expressed two ways using the Commutative Property.

You Try It 13 Apply the idea of identity by filling in the blank. Use the discussion above as a guide.

a) $4 + 0 =$ ______ b) $9 +$ ______ $= 9$ c) $0 +$ ______ $= 12$

d) $7 \cdot 1 =$ ______ e) $23 \cdot$ ______ $= 23$ f) $1 \cdot$ ______ $= 15$

Think About It 2 Is 1 an identity for division? Explain your answer, or use an example to demonstrate your answer.

____

____

Can 0 be an identity for multiplication? No. The product of 0 and any number is always 0.

Think About It 3 The product $a \cdot b$ is 0., What must be the value of the one of the numbers, a or b? Explain your answer.

____

____

We see that 0 is the identity for addition, but not for multiplication. A principle that will occur throughout this text is this:

In general,

The rules of addition are different from the rules of multiplication.

For example, we should not apply the rules of multiplication to a *sum* and expect to get accurate results.

THE DISTRIBUTIVE PROPERTY

It is now time to introduce a property that involves both multiplication and addition in the same expression. It is called the **Distributive Property**. It is called this because, as you will see, we actually distribute one number to two numbers, just as a mail carrier might distribute the same advertisement to two neighbors' houses.

Believe it or not, you have actually used the distributive property quite extensively; maybe more so when you were younger, but you may have come across it relatively recently. Remember the guidelines for multiplying, say, $6 \cdot 17$? Let's look:

6	
x 17	← (17 is 7 + 10)
4 2	← Multiply the 6 times the 7 ($6 \cdot 7 = 42$). Place zero under the two and
+ 6 0	← Multiply 6 times 1 ($6 \cdot 1 = 6$). This multiplication is really $6 \cdot 10 = 60$)
1 0 2	← Add to get the final result.

In algebra, that process might look more like this next example. Look at the similarities between this one is to the previous one (same numbers, different methods).

$6 \cdot 17$	Rewrite 17 as (7 + 10)
$= 6 \cdot (7 + 10)$	Distribute "6 · " through to both numbers;
$= 6 \cdot 7 + 6 \cdot 10$	Multiply 6 times 7 and multiply 6 times 10.
$= 42 + 60$	Add to get the final result
$= 102$	

One might ask if we are now required to multiply two numbers this way. The answer is, "No, not for numerical values." However, in Section 1.9 we will use the Distributive Property to multiply within expressions containing variables.

Here is the Distributive Property in its non-numerical form:

$$b \cdot (c + d) = b \cdot c + b \cdot d$$

or $$b(c + d) = bc + bd.$$

Notice that what is actually being distributed is more than just a number, ***b***; the multiplication symbol is being distributed along with the *b*; so really, "***b*** ·" (*b times*) is being distributed. Also,

The number that is distributed is called the **multiplier**.

We can also distribute when the multiplier is on the right:

$$(c + d)b = cb + db.$$

The formal name for this property is the Distributive Property of Multiplication Over Addition:

The Distributive Property of Multiplication over Addition

1. $\boldsymbol{b(c + d) = bc + bd}$

2. $\boldsymbol{(c + d)b = cb + db}$

Let's practice using the distributive property:

Example 13: Use the distributive property to evaluate each expression.

a) $5(7 + 8)$ b) $(20 + 6)2$ c) $8(5 + 2)$

Answer:

a) $5(7 + 8)$ 5 is the multiplier, and, as there is nothing between it and the quantity, the operation is automatically multiplication.

$= 5\cdot(7 + 8)$ This step isn't necessary, but it shows the multiplication. Distribute "5 times" through to both terms.

$= 5 \cdot 7 + 5 \cdot 8$ Multiply.

$= 35 + 40$ Add

$= 75$

b) $(20 + 6)2$ The multiplier is 2.

$= 20 \cdot 2 + 6 \cdot 2$ This step isn't necessary; it can be done mentally.

$= 40 + 12$

$= 52$

c) $8(5 + 2)$ This time, only the necessary steps are included.

$= 40 + 16$ You can distribute and multiply directly without writing down the multiplication step.

$= 56$

As accurate as the distributive property is, it still leads to some confusion. We've been learning about the order of operations in this section, and the first thing we learn is "Apply the operation inside parentheses first."

When an expression is <u>all numerical</u> the distributive property isn't necessary; in fact, it's rarely used. The distributive property is more important to algebra. Later in the course you'll be re-introduced to the distributive property as it relates to algebra.

For the purposes of practicing this property now, when it's relatively easy, do this next exercise as directed, using the distributive property.

You Try It 14 Use the distributive property to evaluate each. **SHOW ALL STEPS.** (You may *check* your answers using the order of operations.) Use Example 13 as a guide.

a) $5(6 + 3)$ b) $2(8 + 1)$ c) $(10 + 4)6$ d) $(11 + 5)4$

You Try It Answers:

You Try It 1:
a) The sum of five 7's: $7 + 7 + 7 + 7 + 7$
b) The sum of six p's: $p + p + p + p + p + p$
c) Seven factors of 2: $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
d) Three factors of m: $m \cdot m \cdot m$

You Try It 2:
a) $5 \cdot 8 = 40$ b) $9 + 3 = 12$ c) $\sqrt{9} = 3$

You Try It 3:
a) $x \div 3$ b) v^4 c) $p + 10$

You Try It 4:
a) The quotient of 18 and 6.   b) The sum of 25 and a number.
c) The product of 3 and 16.   d) The square root of a number.

You Try It 5:
a) 1 and 12; 2 and 6; 3 and 4   b) 1 and 16; 2 and 8; 4 and 4
c) 1 and 18; 2 and 9; 3 and 6   d) 1 and 20; 2 and 10; 4 and 5

You Try It 6:
a) 6   b) 3   c) 4   d) 14
e) 3   f) 36

You Try It 7:
a) 18   b) 12   c) 4   d) 34
e) 31   f) 2

You Try It 8:
a) 6   b) 2   c) 7   d) 10

You Try It 9:
a) 13   b) 49   c) 80

You Try It 10:
a) 1   b) 2   c) 4   d) 2

You Try It 11:
a) $6 + 2$ b) Subtraction is not commutative.
c) Division is not commutative.   d) $5 \cdot 4$

You Try It 12:
a) $(2 + 8) + 6 = 2 + (8 + 6) = 16$
b) $(6 \cdot 5) \cdot 2 = 6 \cdot (5 \cdot 2) = 60$

You Try It 13:
a) 4   b) 0   c) 12   d) 7
e) 1   f) 15

You Try It 14:
a) 45   b) 18   c) 84   d) 64

Section 1.1 Exercises

Think Again.

1. Can 0 be an identity for subtraction? Explain your answer, or use an example to demonstrate your answer.

2. If an expression with three whole numbers has subtraction as the only operation, does it matter which subtraction we apply first? Show an example to support your answer.

Focus Exercises.

State the meaning of each abbreviation and write it in its expanded form.

3. $3 \cdot 12$ **4.** $7 \cdot w$ **5.** 8^3 **6.** p^5

Expand each and find its value.

7. 3^3 **8.** 2^5 **9.** 11^2 **10.** 4^3

Write each as a mathematical expression and find the result.

11. The quotient of 28 and 4.

12. The square root of 81.

13. The product of 12 and 5.

14. The sum of 19 and 30.

Write each as a mathematical expression.

15. The difference of a number and 17.

16. The product of 9 and a number.

17. The sum of 21 and a number.

18. The third power of a number.

Write each expression in English.

19. $71 - 15$ **20.** $56 \div 8$ **21.** 2^6 **22.** $19 + 65$

23. $w \div 6$ **24.** $\sqrt{y}$ **25.** $p \cdot 11$ **26.** $28 - x$

Use a factor pair table to find all of the factor pairs of:

27. 32 **28.** 40 **29.** 28 **30.** 42

31. 48 **32.** 45 **33.** 60 **34.** 90

Evaluate each according to the order of operations. Show all steps.

35. $30 \div 5 + 1$

36. $(8 + 5) \cdot 2$

37. $5 \cdot 6 \div 3$

38. $5 \cdot 3^2$

39. $2^3 \cdot 3^2$

40. $28 \div 7 \cdot 2$

41. $30 \div 2 \cdot 3$

42. $7^2 + 5 - 3$

43. $(5 \cdot 2)^2 - 7$

44. $(5 + 3) \cdot 9$

45. $6 + [12 \div (2 \cdot 3)]$

46. $24 \div (6 - 2) \cdot 3$

47. $(12 + 28) \div (7 - 3)$

48. $(6 - 2) \cdot 3^2$

49. $9 \cdot \sqrt{25}$

50. $6^2 - \sqrt{25}$

51. $9 + \sqrt{2 \cdot 2}$

52. $3 \cdot \sqrt{9 + 7}$

53. $24 - 8 \div 2 \cdot 4$

54. $8 + 2 \cdot 3^2$

55. $(24 - 8) \div 2 \cdot 4$

56. $(8 + 2) \cdot 3^2$

57. $[9 - (3 + 1)]^2$

58. $5 \cdot 2^2 - 4 \cdot (5 - 2)$

59. $(7 + 3) \cdot (9 + 4)$

60. $(7 + 3)^2 \div (9 - 4)^2$

61. $\dfrac{7 - 4}{3 - 2}$

62. $\dfrac{9 - 1}{6 - 4}$

63. $\dfrac{3 \cdot 9 + 3}{3^2 - 4}$

64. $\dfrac{5^2 + 10}{\sqrt{16} + 3}$

65. $\dfrac{3^3 - \sqrt{49}}{2 \cdot 8 - 6}$

66. $(3)^2 \cdot \dfrac{\sqrt{4}}{(3 - 2)^2}$

Which property does each represent?

67. $5 + (8 + 2) = (5 + 8) + 2$

68. $12 \cdot 5 = 5 \cdot 12$

69. $0 + 11 = 11$

70. $8 \cdot (5 \cdot 4) = (8 \cdot 5) \cdot 4$

71. $25 + 19 = 19 + 25$

72. $13 \cdot 1 = 13$

Think Outside the Box.

Given the numbers 1, 2, 3, and 4, use grouping symbols (if needed) and one or more of the operation symbols (+, –, x, and ÷) to make an expression equal to the underlined number. The numbers may be rearranged in any order, and one or more of the numbers may be used as an exponent.

Example A: if the answer is <u>*18*</u>*, then we can create the expression* $4^2 + 3 - 1 = 18$.

Example B: if the answer is <u>*19*</u>*, then we can create the expression* $(4 + 2) \times 3 + 1 = 19$.

73. <u>11</u>

74. <u>15</u>

75. <u>21</u>

76. <u>45</u>