Section 1.7 The Order of Operations

Objectives
In this section, you will learn to:

- Apply the order of operations to signed numbers.
- Apply exponents to negative numbers.
- Find negative square roots.

To successfully complete this section, you need to understand:

- Exponents (R.1)
- Square roots (R.1)
- Order of operations (1.1)
- Absolute value (1.2)
- Operations with real numbers (1.3, 1.4, 1.5)

INTRODUCTION

In Section 1.1 we discussed the order of operations for positive numbers only. Since that time we have discussed adding, subtracting, multiplying, and dividing signed numbers. This section discusses the order of operations as it applies to both positive and negative numbers.

We were also introduced to another pair of grouping symbols, the *absolute value bars*. Like the square root and the division bar, absolute value bars act as grouping symbols as well as an operation. This means that the absolute value of a number is evaluated before any "outside" operations are applied.

Recall (1.1) that we use *continued equal signs* to show an expression in one step is equivalent to the expression in the next step. When more than one equal sign is needed, the equal signs that follow are aligned directly below the one above it. This helps us keep our work organized and easy to read. We use continued equal signs when evaluating expressions in this section.

EVALUATING EXPRESSIONS

Let's practice using the order of operations with a few exercises.

Example 1: Evaluate each expression according to the order of operations.

a) $-6 + 12 \div 3$ b) $(4 - 16) \div 2^2$

Procedure: Apply the operations one step at a time, from highest rank to lowest rank.

Answer:

a) $-6 + 12 \div 3$ Apply division first.

$= -6 + 4$ Now add.

$= -2$

b) $(4 - 16) \div 2^2$ The parentheses indicate that we must apply subtraction first: $4 - 16 = 4 + (-16) = -12$.

$= -12 \div 2^2$ Now apply the exponent.

$= -12 \div 4$ Now divide.

$= -3$

Example 2: Evaluate each expression according to the order of operations.

a) $\sqrt{24 + 4 \cdot (-2)}$ b) $|\,3^2 - 11\,|$

Procedure: Both the radical and absolute value bars are operations as well as grouping symbols. Because they are grouping symbols, we must evaluate within each before applying the operation.

Answer: a) $\sqrt{24 + 4 \cdot (-2)}$ Within the radical, multiply first: $4 \cdot (-2) = -8$.

$= \sqrt{24 + (-8)}$ Within the radical, now add.

$= \sqrt{16}$ Now apply the square root.

$= 4$

b) $|\,3^2 - 11\,|$ Within the absolute value bars, first apply the exponent.

$= |\,9 - 11\,|$ Within the absolute value bars, now subtract. Note: we could first change $9 - 11$ to $9 + (-11)$ and add. $9 + (-11) = -2$.

$= |\,-2\,|$ Now evaluate the absolute value of -2.

$= 2$

Caution: The key to successfully applying the order of operations is to do one step at a time. It's very important to show your work every step of the way, writing each new step on a new line, as demonstrated in Examples 1 and 2. This will lead to accurate answers and enable others to read and learn from your work.

You Try It 1 Evaluate each expression according to the order of operations. Use Examples 1 and 2 as guides.

a) $-24 \div 6 - 2$ b) $-3 \cdot (5 \cdot 2 - 5)$

c) $|\,-16 + 6 \cdot 2\,|$ d) $\sqrt{(6 - 2) \cdot 5^2}$

EXPONENTS AND NEGATIVE NUMBERS

Recall (1.1) that an exponent is an abbreviation for repeated multiplication. For example, 5^3 means three factors of 5: $5 \cdot 5 \cdot 5 = 125$.

Let's extend this idea to powers (exponents) of negative numbers. For example, we know that $3^2 = 9$. It's also true that $(-3)^2 = 9$. It's easy to demonstrate this using the definition of exponents: $(-3)^2 = (-3)(-3) = 9$.

As we know from Section 1.5, an even number of negative factors has a product that is positive. So, when a base is negative and its exponent is an *even* number, the end result will be a positive number. For example,

a) $(-5)^2 = (-5)(-5) = +25$ (two negative factors)

b) $(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = +64$ (six negative factors)

c) $(-10)^4 = (-10)(-10)(-10)(-10) = +10{,}000$ (four negative factors)

Likewise, an odd number of negative factors has a product that is negative. So, when a base is negative and its exponent is an *odd* number, the end result will be a negative number. For example,

a) $(-2)^5 = -32$, b) $(-5)^3 = -125$, and c) $(-10)^7 = -10{,}000{,}000$.

In general: $(\text{Negative base})^{\text{even exponent}} = +$ number

$(\text{Negative base})^{\text{odd exponent}} = -$ number

Think About It 1 Will the result of $(-6)^5$ be positive or negative? Explain your answer.

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Think About It 2 Will the result of $(9)^7$ be positive or negative? Explain your answer.

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Example 3: Evaluate each expression.

a) $(-9)^2$ b) $(-3)^3$ c) $(-2)^4$ d) $(3 - 5)^3$ e) $(-8 + 3)^2$

Procedure: For parts a), b), and c), first determine if the result is going to be positive or negative, based on the exponent. For parts d) and e), first evaluate within the grouping symbols and then decide whether the result will be positive or negative.

Answer:

a) $(-9)^2 = 81$ The exponent is 2, an even number, so the result is positive; $9 \cdot 9 = 81$.

b) $(-3)^3 = -27$ The exponent is 3 (odd), so the result is negative; $3 \cdot 3 \cdot 3 = 27$.

c) $(-2)^4 = +16 = 16$ The exponent is 4 (even), so the result is positive; $2 \cdot 2 \cdot 2 \cdot 2 = 16$.

d) $(3 - 5)^3 = (-2)^3 = -8$ The exponent is 3 (odd), so the result is negative; $2 \cdot 2 \cdot 2 = 8$.

e) $(-8 + 3)^2 = (-5)^2 = 25$ The exponent is 2 (even), so the result is positive; $5 \cdot 5 = 25$.

You Try It 2 Evaluate each expression. Use Example 3 as a guide.

a) $(-7)^2$ b) $(-4)^3$ c) $(-1)^5$ d) $(1 - 9)^2$ e) $(-9 - 1)^3$

Recall (1.1) that grouping symbols create a quantity (a single value). When we see the expression $(-3)^2$, we understand that, because of the parentheses, the negative sign is included with the 3 in the squaring:

$$(-3)^2 = (-3)(-3) = +9.$$

However, if there are no parentheses, as in -3^2, the negative is not included in the squaring. To understand this, recall (1.5) that one meaning of the dash, the negative sign, is "the opposite of." This means that we can interpret -3^2 as "the opposite of 3^2," which is *the opposite of* 9, or -9.

Another interpretation of the dash is "-1 times." If we write the dash using this interpretation, -3^2 becomes $-1 \cdot 3^2$. Then the order of operations tells us to apply the exponent first and then multiply:

$$-1 \cdot 3^2 = -1 \cdot 9 = -9$$

Notice that this expression contains just one negative sign, so the result is negative. With parentheses, $(-3)^2$ contains two negative signs, so the result is positive. Even if we wrote -3 as $-1 \cdot 3$, with parentheses we still get two negative signs:

$$(-1 \cdot 3)^2 = (-1 \cdot 3)(-1 \cdot 3) = (-3)(-3) = +9$$

The result of this discussion is this:

> If a is a positive number, then $-a^2 = -1 \cdot a^2$ is a negative number.
>
> If there are no parentheses around the base, then any negative sign is not included in the squaring of the base.

Example 4: Evaluate each expression.

a) -5^2 b) $-6^2 + 10$ c) $(-4)^2 - 2^2$ d) $-3^2 - 2^2$

Procedure: If there are no parentheses around the base, then the negative is not included in the squaring.

Answer:

a) -5^2 — There are no parentheses around the base, 5, so we square only 5. The result is negative.

$= -25$

b) $-6^2 + 10$ — There are no parentheses around the base, 6, so -6^2 is -36.

$= -36 + 10$ — Now add.

$= -26$

c) $(-4)^2 - 2^2$ — There *are* parentheses around the -4, so $(-4)^2$ is +16.

$= 16 - 2^2$ — 2^2 is 4.

$= 16 - 4$ — Now subtract.

$= 12$

d) $-3^2 - 2^2$ — There are no parentheses around the first base, 3, so -3^2 is -9

$= -9 - 2^2$ — 2^2 is 4.

$= -9 - 4$ — Now subtract; or, add the opposite, $-9 + (-4) = -13$.

$= -13$

You Try It 3 Evaluate each expression. Use Example 4 as a guide.

a) -8^2 b) $(-10)^2$ c) -7^2 d) -6^2

e) $(-9)^2$ f) $-1^2 - 7$ g) $(-7)^2 - 3^2$ h) $-5^2 - 4^2$

NEGATIVE SQUARE ROOTS

We have seen that both $3^2 = 9$ and $(-3)^2 = 9$. This example demonstrates that there are two numbers for which 9 is a perfect square. It's appropriate, therefore, to say that 9 has *two* square roots:

1. A positive square root, +3, and
2. a negative square root, -3.

The Square Root:

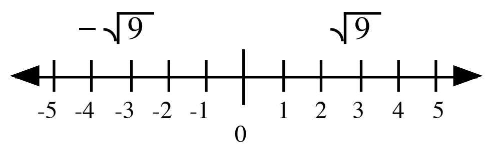
If $r^2 = P$ then r is a square root of P.

Does this change what we've learned about the square root radical? No, $\sqrt{9}$ is still *the square root of 9*, but now we emphasize that it refers (as it always has) only to the positive square root: $\sqrt{9} = +3$, but not -3.

We call the positive square root of a number the **principal square root**.

If we wish to use the radical to represent the negative square root of 9, we must place a negative sign before (outside of) the radical: $-\sqrt{9} = -3$.

Here are the positive and negative square roots of 9 represented on the number line.



Think About It 3 What is $\sqrt{0}$? Explain your answer.

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What about $\sqrt{-9}$? Is there any number, a, that can be squared to equal -9? a^2 must be positive, so there is no real number whose square is a negative number. This means that $\sqrt{-9}$ is undefined in the real number system. This is true for the square root of every negative number.

Example 5:	Evaluate each radical expression, *if possible*. If the radicand is negative, write "undefined".			
	a) $\sqrt{36}$	b) $-\sqrt{36}$	c) $\sqrt{-36}$	d) $-\sqrt{-36}$
Procedure:	Notice the placement of any negative sign in the expression. If a negative sign is within the radical, then the expression is undefined.			
Answer:	a) 6	b) -6	c) undefined	d) undefined

You Try It 4 Evaluate each radical expression, *if possible*. If the radicand is negative, write "undefined". Use Example 5 as a guide.

a) $-\sqrt{100}$ b) $-\sqrt{-64}$ c) $\sqrt{-1}$ d) $-\sqrt{4}$

DOUBLE QUANTITIES

Recall (1.1) that if an expression contains two sets of grouping symbols, we can evaluate within each of them at the same time. Also recall (1.1) that the division bar is a grouping symbol; it groups the numerator and denominator separately.

Example 6: Evaluate each according to the *order of operations*.

a) $(24 \div 6) - \sqrt{5+11}$ b) $|4-6| - |-7+4|$

c) $\dfrac{\sqrt{16}+5}{2^2-7}$ d) $30 \div \dfrac{\sqrt{36}}{8-5}$

Procedure: Single operations within different grouping symbols (including the numerator and the denominator) are separate quantities, and they can be applied at the same time.

Answer:

a) $(24 \div 6) - \sqrt{5+11}$ We can apply both the division and the addition in the same step.

$= 4 - \sqrt{16}$ Now apply the radical.

$= 4 - 4$ Now subtract.

$= 0$

b) $|4-6| - |-7+4|$ Evaluate within each set of absolute value bars: $|4-6| = |-2|$ and $|-7+4| = |-3|$.

$= |-2| - |-3|$ Now we can apply each absolute value, not as a grouping symbol but as an operation.

$= 2 - 3$ Now subtract. We change subtraction to adding the opposite: $2 + (-3) = -1$

$= -1$

c)	$\dfrac{\sqrt{16} + 5}{2^2 - 7}$	First apply the radical in the numerator and square the 2 in the denominator.
	$= \dfrac{4 + 5}{4 - 7}$	Next, separately apply subtraction and addition.
	$= \dfrac{9}{-3}$	Now divide: $9 \div (-3) = -3$.
	$= -3$	
d)	$30 \div \dfrac{\sqrt{36}}{8 - 5}$	Simplify the fraction completely before applying the first division, ÷. Apply the radical in the numerator and subtract in the denominator.
	$= 30 \div \dfrac{6}{3}$	Next simplify the fraction using division: $6 \div 3 = 2$.
	$= 30 \div 2$	Now divide.
	$= 15$	

You Try It 5 Evaluate each expression according to the order of operations. Use Example 6 as a guide.

a) $(6 + 3)^2 - \sqrt{-4 + 5}$

b) $|7 - 4| - |-10|$

c) $\dfrac{-4 - 12}{3^2 - 1}$

d) $\dfrac{\sqrt{9} - 15}{-4 - (-2)}$

REPLACEMENT VALUES AND FORMULAS

Many formulas—in a variety of fields—require the order of operations for their proper evaluation. For example, if you need to convert a temperature reading from Celsius degrees to Fahrenheit degrees, you can use the formula

$$F = \frac{9}{5}C + 32, \qquad \text{where:} \quad F = \text{Fahrenheit degrees}, \quad C = \text{Celsius degrees}$$

You may have seen Celsius temperatures, for example, at bank buildings. The sign out front might say that the temperature is $15° \ C$, but is that t-shirt or jacket weather?

To answer that question, we must *substitute* the known value of C (15°) into the formula. We then evaluate the expression on the right side using the order of operations.

> To **substitute** a *value* into a formula means to *replace* the variable with a number; this number is called the **replacement value**. Whichever operation was applied to the variable is now applied to the replacement value.

When we see $F = \frac{9}{5} C + 32$, $\mathbf{C = 15}$ ← This means we replace C with 15.

$F = \frac{9}{5}(15) + 32$ Write 15 as a fraction, $\frac{15}{1}$.

$F = \frac{9}{5} \cdot \frac{15}{1} + 32$ Now multiply the fractions and simplify by a factor of 5: $\frac{9}{5} \cdot \frac{15}{1} = \frac{9}{1} \cdot \frac{3}{1} = \frac{9 \cdot 3}{1 \cdot 1}$

$F = \frac{9 \cdot 3}{1 \cdot 1} + 32$ Use the order of operations to complete the evaluation.

$F = 27 + 32$

$F = 59$

We can interpret this as saying $15° \ C$ is equivalent to $59° \ F$. By the way, you might want to put on a jacket; it's starting to get a little chilly!

You may have noticed the use of parentheses around the 15. Here, the parentheses are used more as a separator than as grouping symbols. When substituting a value into a formula, it is usually proper to put parentheses around that number, especially when

1) the variable is being multiplied or divided; or

2) The replacement value is a negative number—we must make sure that the negative sign is grouped with the number.

Many formulas have more than one variable that must be replaced. In this case, substitute for all such variables—using their corresponding replacement values—at the same time. Be especially careful in your use of parentheses.

Example 7: Evaluate each formula based on the given replacement values.

a) $d = R \cdot t$ $\quad R = 50$, $t = 2$

b) $E = \dfrac{9R}{I}$ $\quad R = 10$, $I = 45$

c) $m = \dfrac{y - p}{x - q}$ $\quad y = -2$, $p = 4$, $x = 6$, $q = -3$

d) $c = \sqrt{a^2 + b^2}$ $\quad a = 3$, $b = 4$

Procedure: Place each given value into the formula and evaluate using the order of operations.

Answer:

a)
$$d = R \cdot t$$
$$d = (50)\cdot(2)$$
$$d = \boxed{\mathbf{100}}$$

b)
$$E = \frac{9R}{I}$$
$$E = \frac{9(10)}{45}$$
$$E = \frac{90}{45} = \boxed{\mathbf{2}}$$

c)
$$m = \frac{(-2) - 4}{6 - (-3)}$$
$$m = \frac{-6}{6 + 3}$$
$$m = \frac{-6}{9}$$
$$m = \boxed{-\frac{\mathbf{2}}{\mathbf{3}}}$$

d)
$$c = \sqrt{a^2 + b^2}$$
$$c = \sqrt{3^2 + 4^2}$$
$$c = \sqrt{9 + 16}$$
$$c = \sqrt{25} = \boxed{\mathbf{5}}$$

You Try It 6 Evaluate each formula based on the given replacement values. Use Example 7 as a guide.

a) $d = R \cdot t$ $R = 15$ $t = 3$

b) $A = \dfrac{a + b + c}{3}$ $a = 15$ $b = 23$ $c = 34$

c) $m = \dfrac{y - p}{x - q}$ $y = 10$ $p = -6$ $x = -5$ $q = 7$

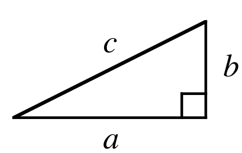
d) $z = \dfrac{x - m}{s}$ $x = 24$ $m = 16$ $s = 4$

Where do we get all of these formulas from, anyway? In Example 7 and You Try It 6 we saw six formulas from a variety of disciplines or interests:

(a)	$d = r \cdot t$	from physics:	distance = rate · time
(b)	$E = \dfrac{9R}{I}$	from baseball:	a pitcher's "earned run average" R = # runs allowed; I = # innings pitched
(c)	$m = \dfrac{y - p}{x - q}$	from algebra:	the slope of a line
(d)	$A = \dfrac{a + b + c}{3}$	from statistics:	the average of three numbers
(e)	$z = \dfrac{x - m}{s}$	from statistics:	an important conversion formula
(f)	$c = \sqrt{a^2 + b^2}$	from geometry:	the Pythagorean Theorem

c = the length of the longest side of a right triangle and

a and b are the lengths of the other two sides



You Try It Answers

(Note: Positive answers may also be written with a plus sign in front.)

You Try It 1:	a) -6	b) -15	c) 4	d) 10
You Try It 2:	a) 49	b) -64	c) -1	
	d) 64	e) -1,000		
You Try It 3:	a) -64	b) 100	c) -49	d) -36
	e) 81	f) -8	g) 40	h) -41
You Try It 4:	a) -10	b) Undefined	c) Undefined	d) -2
You Try It 5:	a) 80	b) -7	c) -2	d) 6
You Try It 6:	a) $d = 45$	b) $A = 24$	c) $m = -\frac{16}{12} = -\frac{4}{3}$	d) $z = 2$

Section 1.7 Exercises

Think Again.

1. 30 is not a perfect square, so it does not have a whole number square root. Between which two consecutive integers is the value of $-\sqrt{30}$? Explain your answer or show an example that supports your answer.

2. If a is a negative number, then is a^2 positive or negative? Explain your answer or show an example that supports your answer.

Focus Exercises.

Evaluate.

3. $(-1)^2$	**4.** $(-4)^1$	**5.** -4^2	**6.** $(-6)^2$
7. -2^2	**8.** $(-2)^3$	**9.** $\sqrt{64}$	**10.** $\sqrt{49}$
11. $-\sqrt{81}$	**12.** $-\sqrt{25}$	**13.** $\sqrt{-4}$	**14.** $-\sqrt{-100}$

Evaluate each expression using the order of operations.

15. $-5 + 6 \cdot 2$

16. $6 \cdot 2 - 4 \cdot 8$

17. $-24 \div 6 \cdot (-2)$

18. $|6| - 8$

19. $9 - |-8|$

20. $|9 - 5|$

21. $3 - 4 \cdot 5$

22. $-24 \div (3 - 5)$

23. $(3 - 10)^2$

24. $-9 + (2 - 5) + 4^2$

25. $-3(2 - 8)$

26. $(-4 + 10) \div (-3)$

27. $(7 - 9)^3 - 6 \cdot 3$

28. $[-12 \div (-3)]^2$

29. $-10 \cdot \sqrt{64}$

30. $-6 - 3\sqrt{4}$

31. $\sqrt{80 \div 5} - 9$

32. $(5 - 9)^2$

33. $(4 - 10)^2$

34. $(4 - 6)^3$

35. $(7 + 3) \cdot (-9 + 4)$

36. $|3 - 9| + 8$

37. $|-3| + |8|$

38. $|11| - |-15|$

39. $|3 + 7| + |4 - 2|$

40. $|5 - 9| - |2 - 8|$

41. $18 - 2 \cdot \sqrt{100}$

42. $\sqrt{-1 + 2 \cdot 5}$

43. $-\sqrt{5^2 - 3^2}$

44. $\dfrac{7 - 4}{2 - 3}$

45. $\dfrac{-1 - 9}{-6 - 4}$

46. $\dfrac{-6 - 8}{5 - (-1)}$

47. $\dfrac{-5 \cdot 2 + 8}{4 - 2^3}$

48. $(7 + 3)^2 \div (9 - 4)^2$

49. $(5 - 6)^3 - 4 \cdot (-2)$

50. $35 \div (-5) + 2 \cdot (-3)$

Evaluate the numerical value of each formula with the given replacement values.

51. $z = \dfrac{x - m}{s}$ $x = 16$, $m = 25$, $s = 3$

52. $A = \dfrac{a + b + c}{3}$ $a = 13$, $b = 41$, $c = 33$

53. $a = \sqrt{c^2 - b^2}$ $c = 13$, $b = 12$

54. $P = 2 \cdot \mathrm{L} + 2 \cdot W$ $\mathrm{L} = 13$, $W = 8$

55. $A = \frac{1}{2} \cdot h \cdot (b + B)$ $\quad h = 3$, $b = 5$, $B = 7$

56. $m = \frac{y - w}{x - v}$ $\quad y = 1$, $w = -8$, $x = -6$, $v = -3$

Think Outside the Box.

Evaluate.

57. $\sqrt{-5 + 9}^{\,4}$

58. $|-7 + 4|^{|-7 + 4|}$

59. $(-6 + 10 \div 2)^{(-7 + 10)}$

60. $(-8 + 6)^{(-2)^2}$