### 1.1 The Real Number System

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## Number Lines

When you count you usually begin with 1 , then go to 2 , then to 3 , etc. For this reason, those numbers are called counting numbers, or natural numbers; they are also referred to as whole numbers. However, 0 is also a whole number, but it is not a counting number since we never start counting from 0 .

| The List of Counting (Natural) Numbers: | $1,2,3,4,5, \ldots$ |
| :--- | ---: |
| The List of Whole Numbers: | $0,1,2,3,4,5, \ldots$ |

e can represent numbers visually along a number line.


The number zero (0) has a special name on a number line; it is called the origin, which means the beginning.

A six-inch ruler is a good example of a horizontal number line.

|  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | - 2 | 23 | 4 | 5 |



There are also vertical number lines; one is a thermometer.
An outdoor thermometer (at left) includes numbers less than zero to indicate temperature "below zero."

This thermometer has two temperature scales on it: $\mathbf{C}$ for Celsius and F for Fahrenheit.

These numbers less than 0 are called negative numbers. Numbers more than (greater than) 0 are called positive numbers.

On a VERTICAL number line (at right), 0 is still the origin, the positive numbers are above 0 and the negative numbers are below 0 .

On a HORIZONTAL number line positive numbers are to the right of 0 and negative numbers are to the left of 0 . Even though 0 is in the middle now, it is still referred to as the origin.


Notice that negative numbers have a "minus" sign-a dash—in front of them. For the sake of clarity, we sometimes put a plus sign in front of positive numbers, such as +5 . You'll see many examples of that throughout this chapter.

Numbers on different sides of zero, but the same distance away, are called opposites of each other. Numbers such +2 and -2 are opposites of each other. They are on opposite sides of 0 and are the same distance away from 0 . Lastly, the opposite of 0 is itself.

Example 1: Fill in the blank. (The answers are given directly.)
a) The opposite of +15 is -15
b) The opposite of - 16 is +16
c) $\quad-20$ is the opposite of +20
d) +23 is the opposite of -23

## $\overline{\text { Exercise 1: }} \quad$ Fill in the blank.

a) The opposite of 6 is
c) The opposite of - 1 is $\qquad$
e) __ is the opposite of 13
g) __ is the opposite of - 3
i) 7 is $\quad-7$
k) -2 is $\qquad$ 2
b) The opposite of -9 is $\qquad$
d) The opposite of 12 is $\qquad$
f) $\quad$ is the opposite of -10
h) __ is the opposite of 5
j) $\quad-8$ is $\quad 8$

1) $\quad 4$ is $\quad-4$

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## Absolute Value

A relatively quiet, unassuming notion that generally follows the number line is the absolute value of a number.

The absolute value of a number is the distance that number is-along the number line-from the origin, 0 .

As mentioned above, a number and its opposite are on different sides of 0 , on the number line, and are the same distance from 0, the same number of units away from 0 , so they must have the same absolute value.

For example, the absolute value of 3 is the same as the absolute value of -3 because they are the same distance away from 0 . So what is their absolute value? The key word in the definition is distance. Distance is always measured as a positive number.

Even if someone walks backwards for 20 feet, the distance traveled is still +20 feet, meaning direction doesn't matter, but distance does. In this sense, the absolute value of any number is always positive.

The absolute value of 3 is $\mathbf{3}(+3)$.
The absolute value of -3 is also $3(+3)$.

Of course, this being algebra, there is a symbol that is used to indicate the absolute value of a number. Actually, it consists of two symbols working together. They are called "absolute value bars" and look like this: $\mid$ and $\mid$. So, we could write the "the absolute value of 3 is $\mathbf{3}$ " as $|3|=3$; similarly, "the absolute value of -3 is $\mathbf{3}$ " could be written as $|-3|=3$.

Notice that the absolute value maintains positivity for positive numbers but it becomes the opposite for negative numbers. DO NOT CONFUSE THE "ABSOLUTE VALUE" WITH THE "OPPOSITE." The absolute value of a positive number is not its opposite.

$$
\text { Example 2: } \quad \begin{array}{lll}
|9|=9 & 9 \text { is a distance of nine units away from the origin. } \\
|-7|=7 & -7 \text { is a distance of seven units away from the origin. } \\
|0|=0 & 0 \text { is no distance away from itself. }
\end{array}
$$

Exercise 2: Find the absolute value of each number, as indicated.
a) $\quad|12|=$
b) $|-5|=$ $\qquad$
c) $\quad|24|=$ $\qquad$
d) $|-1|=$ $\qquad$
e) $\quad|0|=$ $\qquad$ f) $|-4|=$ $\qquad$

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## A New Definition of Number

Now that you have been introduced to negative numbers, it's time to think of numbers as having both a value and a direction, left or right (of zero).

Here is a new definition of a number. This idea is extremely helpful in understanding how positive and negative numbers work together.

## Numbers have both a numerical value and a direction.

- The numerical value is its distance, along the number line, from 0; the numerical value is another name for the absolute value.
- The direction is either left or right, depending on its location on the number line when compared to 0 (zero):
- Negative numbers are to the left and positive numbers are to the right.

For the sake of simplicity, I'll revise the new definition of numbers ever so slightly

## Every number has both value and direction.

We will use this new definition extensively in Section 1.2 when we discuss adding positive and negative numbers. Positive and negative numbers are also called signed numbers.

Example 3: Identify the (numerical) value and direction of each of the numbers.
a) 6 The value is 6 and the direction is to the right.
b) - $8 \quad$ The value is 8 and the direction is to the left.
c) $\quad 0 \quad$ The value is 0 and it has no direction.
d) $+9 \quad$ The value is 9 and the direction is to the right.
$\overline{\text { Exercise 3: }}$ Identify the (numerical) value and direction of each of the numbers. Fill in the blanks.
a) 3 The value is $\qquad$ , and the direction is $\qquad$
b) - 5 The value is $\qquad$ , and the direction is $\qquad$
c) +7 The $\qquad$ is $\qquad$ , and the direction is $\qquad$
d) - 4 The $\qquad$ is $\qquad$ , and the $\qquad$ is $\qquad$
e) +2 The $\qquad$
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## The Real Numbers

Every number that you can think of is a real number, and it has a place somewhere on the number line. Here is a piece of the real number line. 0 (zero) is in the very middle; positive numbers are to the right, and negative numbers are to the left.


At the far right is the symbol for infinity, $\infty$, and at the far left is the symbol for "negative" infinity. These infinities suggest that the number line goes on in each direction indefinitely.

## The Natural Numbers

Within the set of real numbers are many "subsets" (smaller sets of numbers within the larger set) that we can identify. The simplest subset of familiar numbers is, as we have already seen, the set of natural numbers, or "counting numbers."

## The Set of Natural Numbers (Counting Numbers) $\quad\{1,2,3,4,5,6,7, \ldots\}$

(The three dots, called an "ellipsis," indicates the list pattern continues indefinitely.)
Here is the number line for the set of natural numbers:


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## The Whole Numbers

Today, we recognize the number zero as being very useful in our numbering system, but it clearly is not a counting number. In fact, the concept of 0 being represented as its own number came later in the development of most number systems. This leads to a slightly modified set of numbers called the Whole Numbers:

The Set of Whole Numbers (Natural numbers plus zero) $\quad\{0,1,2,3,4,5, \ldots$.

Here is the number line for the set of whole numbers:


## The Integers

Certainly in any society where goods or services exchange hands, whether it be trading loaves of bread for eggs or trading coins for services, the concepts of borrowing, owing, and being in debt quickly occur. At this point, any numbering system would need to include symbols which would indicate debt. Hence, the negative (whole) numbers were created. Along with whole numbers $(0,1,2, \ldots$.$) we now have a set$ called the Integers:

Integers Includes all whole numbers along with all negative natural numbers; we often times say "it is the set of all positive and negative whole numbers." They can be written $\{\ldots . .-3,-2,-1,0,1,2,3, \ldots \ldots$.

0 (zero) is an integer but is neither positive nor negative.

This number line represents the integers:


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The Rational Numbers

We have already been introduced to the number line, and-as you probably know-there are more than just integers on the number line; we can also locate and represent fractions and decimals, both positive and negative. Let's take a closer look at the number line so that we can see how some of the fractions can be located:


Of course, there are many more fractions between 0 and 1 , and there are many more fractions elsewhere on the number line, both positive and negative fractions. Fractions, both positive and negative, are called rational numbers. The word rational is used because it has as its root the smaller word "ratio". A ratio-and a fraction-is a comparison between two numbers by division.

Every number that can be written as fraction-in which both the numerator and denominator are integers-is a rational number.

However, the denominator may never be 0 (zero).

Example 4: Each of these numbers is a rational number. State why.
a) $\frac{1}{2}$
b) $-\frac{3}{4}$
c) 6
d) . 13
e) 0

Procedure: See how it is possible to make each number a fraction.

Answer:
a) $\quad \frac{1}{2}$ is already a fraction, so it is automatically a rational number.
b) $\quad-\frac{3}{4}$ is a fraction; even though it is negative it is still rational.
c) 6 can be written as $\frac{6}{1}$, so it is a rational number.
d) .13 can be written as $\frac{13}{100}$, so it is also a rational number.
e) Like 6,0 can be written as $\frac{0}{1}$, so it is a rational number.
$\overline{\text { Exercise 4: }} \quad$ State why each of these numbers is a rational number.
a) $\frac{4}{5}$
b) -8
c) .4
d) 7
e) $-\frac{8}{3}$
f) 1.35

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## Decimals as Rational Numbers

Any decimal that can be written as a fraction is also considered to be a rational number. For example, 3.68 can be written as $\frac{368}{100}$, and 0.045 can be written as $\frac{45}{1,000}$. These types of decimals are called terminating decimals because they stop, or terminate, after a few decimal places.

Example 5: Write each of these decimal numbers as fractions.
a) $\quad .7$
b) $\quad 9.4$
c) -. 005

Procedure: The number of decimal places determines the power of 10 in the denominator.
Answer: a) $\quad .7=\frac{7}{10}$, thereby making it a rational number.
b) $\quad 9.4=\frac{94}{10}$, thereby making it a rational number.
c) $-.005=\frac{-5}{1,000}$, thereby making it a rational number.
$\overline{\text { Exercise 5: }}$ Write each decimal number as a fraction, thereby making it a rational number.
a) $.9=$ $\qquad$
b) $\quad 2.5=$
c) $\quad 0.079=$
e) $\quad-6.07=$
d) $-.3004=$
d) $-.3004=$
f) $.00009=$ $\qquad$

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Repeating decimals are decimals that never end but have a specific repeated sequence of digits. Examples of repeating decimals are below. Notice that repeating decimals can be abbreviated by using a bar over the repeating pattern. Anything that precedes the repeating pattern is not included under the bar.
2.88888888888888... (the 8 repeats indefinitely.) It can be written as $2 . \overline{8}$
$0.56565656565656 \ldots$ (the 56 repeats indefinitely.) It can be written as $0 . \overline{56}$
$-4.2176363636363 \ldots$ (the 63 repeats indefinitely.) It can be written as $-4.217 \overline{63}$
$3.208152081520815 \ldots$ (the 20815 repeats indefinitely.) It can be written as $3 . \overline{20815}$

Example 6: Abbreviate each repeating decimal by placing a bar over the repeating pattern. Be sure to exclude, under the bar, any number that is not in the pattern. (Look carefully for where the repeating pattern starts and ends.)
a) $\quad .77777777 \ldots=. \overline{7}$
b) $\quad 9.434343434343 \ldots=9 . \overline{43}$
c) $\quad 0.408252525 \ldots=0.408 \overline{25}$
d) $-.351206206206206 \ldots=-.351 \overline{206}$
$\overline{\text { Exercise 6: Abbreviate each repeating decimal by placing a bar over the repeating pattern. Be sure }}$ to exclude, under the bar, any number that is not in the pattern.
a) . $55555555 \ldots=$ $\qquad$
b) $.344444444 . . .=$ $\qquad$
c) $52.383838 \ldots=$ $\qquad$
d) $-.257918404040 \ldots=$ $\qquad$
e) $-3.05291291 \ldots=$ $\qquad$
f) $.216892168921689 \ldots=$ $\qquad$
Decimals that are rational numbers-that can be written as fractions-must follow strict guidelines; such decimals are either
(1) terminating decimals, or
(2) repeating decimals.

So, any number that can be written as a fraction-with integers in both the numerator and denominator-is a rational number. (Of course, the denominator may never be 0 .) Some examples of rational numbers follow.

Example 7: Justify why each is a rational number:
a) $\frac{-3}{5} \quad$ is a rational number, since -3 and 5 are both integers.
b) $0.3333 \ldots$ is a rational number, since it is a repeating decimal.
c) 0.167 is a rational number, since it terminates.
d) 6 is a rational number, since the integer 6 can be expressed as the fraction $\frac{6}{1}$.
$\overline{\text { Exercise 7: }} \quad$ Justify why each is a rational number. Write it out. Use Example 7 as a guide.
a) - 5
b) 2.0815
c) $3.6 \overline{21}$
d) $\frac{6}{-5}$
e) 12
f) $0.89555555 \ldots$
g) $4 \frac{2}{3}$

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## Writing (SOME) Repeating Decimals as Fractions

As has been stated, repeating decimals can be written as fractions. If a decimal has nothing but repeating digits, then we can use a simple method to convert that repeating decimal into a fraction. For example, repeating decimals such as the following can be easily written as fractions (the repeating pattern is alternately highlighted for easy inspection.

$$
\begin{aligned}
.7777777777 \ldots & =. \overline{7} \\
.5454545454 \ldots & =. \overline{54} \\
.238238238238238 \ldots & =. \overline{238}
\end{aligned}
$$

The simple method is quite similar to writing terminating decimals as fractions. We know that terminating decimals can be written as fractions by counting the number of decimal places and making sure there are that many 0 's in the denominator:

$$
.7=\frac{7}{10} \quad \text { and } \quad .54=\frac{54}{100} \quad \text { and } \quad .238=\frac{238}{1,000}
$$

The easy method for writing repeating decimals as fractions is similar to those terminating decimals, above; for repeating decimals, though, we count the number of decimal places and put that many 9's in the denominator. The repeated pattern fits in the numerator. This is demonstrated in the next example.

Example 8: Write each repeating decimal as a fraction; reduce the fraction if possible.
a) .777777....
b).$\overline{43}$
c) . $\mathbf{2 3 8 2 3 8 2 3 8 2 3 8} \ldots$
d).$\overline{3}$
e) .54545454....
f).$\overline{675}$

Procedure: Count the number of digits in the repeated pattern. Write a fraction with the repeated sequence as the numerator and the denominator as a sequence of 9 's. There will be as many 9's in the denominator as there are digits in the repeated sequence.
(The first three cannot be simplified, but the last three can.)
Answer:


Exercise 8: Write each repeating decimal as a fraction; reduce the fraction if possible.
a) . $4444444 \ldots$
b).$\overline{6}$
c) $.26262626 . \ldots$.
d).$\overline{72}$
e) . $524524524 \ldots .$.
f).$\overline{315}$

If a decimal has other digits before the repeating pattern, then a different method, more challenging than the one presented above, would need to be used. We would need to use the more challenging method for decimals such as the following because they contain more than just the repeating pattern.

Each of these would require a different method, one not presented in this book:

$$
\begin{array}{lll}
.23 \overline{8} & .00 \overline{2} & .1 \overline{3429}
\end{array}
$$

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## The Irrational Numbers

Let's look at the magnified number line again.


Of course, between the fractions shown there are even more fractions; there are also many decimals, some of which are rational, some of which are not.

Those numbers that are not rational are called ir - rational numbers. An irrational number is one that cannot be written as a fraction; in other words, an irrational number is a "non-terminating, nonrepeating" decimal.


One example of an irrational number is: $2.15642328493078425920035105 \ldots$
Notice that this does not repeat and it does not terminate. There clearly is no pattern of repetition, so the ellipsis at the end simply means that it continues randomly indefinitely.

Every radical (square root) of a non-perfect square number is irrational.

This means that numbers like $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$, and $\sqrt{10}$, etc. are all irrational numbers.

For example, $\sqrt{2}$ is approximately $1.4142135623730590488016 \ldots$. Nowhere does it repeat for more than two digits at a time. In this case, the ellipsis at the end means that it continues on indefinitely without any pattern at all.

Another irrational number, one that is certainly more interesting and more commonly used, is the number $\pi$ (the Greek letter "pi"). It is a number that compares the circumference (the outer rim) of a circle to its diameter-it's a ratio, but this ratio is a decimal that never ends and never repeats any pattern, so it is irrational.

The circumference is the distance around the circle, as if an ant were walking its entire path. The diameter is straight across the circle, through its center point. If they could be precisely measured, we'd find that

$$
\pi=\frac{\text { Circumference }}{\text { diameter }}=3.1415926535898 \ldots
$$



This is irrational, and, even though it is a fraction, it isn't a fraction made up entirely of integers; in other words, if the diameter is an integer, it turns out that the circumference isn't, and if the circumference is an integer, the diameter isn't.

Lastly, real numbers are all the numbers that have been discussed in this section! Every number we deal with is a real number. Our whole numbering system is based on the real numbers and has a hierarchy that look like this:

The Real Number System


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## Undefined Values

There are actually "numbers" that are not in the real number system. As amazing as that may be, we need not worry about those "numbers" in this course, and forget that it was even mentioned (like telling a jury to ignore the last remark from an attorney).

However, we may come across some "numbers" that are undefined values in the real number system; in other words, they do not exist in the real number system. An example of an undefined value is a fraction with zero (0) as the denominator:
$\frac{\mathrm{a}}{\overline{0}}$ is undefined (or is an "undefined value"). Zero can never be in the denominator But, $\frac{0}{\mathrm{~b}}=0$. Zero can be in the numerator, as long as the denominator is not also 0 . $\frac{8}{0}$ is undefined, and $\frac{0}{0}$ is undefined, but $\frac{0}{7}=0$ is a real number.

Many people get confused about 0 within a fraction. To better understand why 0 in the numerator is okay and has the value 0 , consider that any fraction $\frac{A}{\bar{B}}$ can be written as $A \cdot \frac{1}{\bar{B}}$.

So, think of $\frac{0}{7}$ as $0 \cdot \frac{1}{7}=0$, since 0 times any number is zero.

| Example 9: <br> a) <br> b) <br> c) <br> d) <br> e) <br> f) <br> g) <br> h) <br> i) | Decide into wh natural, whole undefined, then category. List $\frac{6}{11}$ <br> - 9 <br> 18 <br> $\frac{0}{4}=0$ <br> $\frac{-6}{0}$ <br> $\frac{0}{0}$ <br> 0.429731651.. <br> 2.161616... <br> 7.125 | category or set each real number fits. Choose your answers from integer, rational, irrational or undefined. (If the number is is not a real number.) Some numbers will fit into more than one that apply. <br> is rational (both the numerator and denominator are integers) <br> is an integer and rational <br> is natural, whole, an integer and rational <br> is whole, an integer and rational <br> is undefined (0 may never be in the denominator) <br> is undefined (even still, 0 may never be in the denominator) <br> is irrational (a non-repeating and non-terminating decimal) <br> is rational <br> (a repeating decimal) <br> is rational (a terminating decimal) |
| :---: | :---: | :---: |

$\overline{\text { Exercise 9: }}$ Decide into which category or set of the real numbers each fits. Choose your answers from natural, whole, integer, rational, irrational or undefined. (By the way, if the number is undefined, then it is not a real number.)
a) 7
c) $\frac{-3}{8}$ $\qquad$
e) $\sqrt{12}$ $\qquad$
g) 4.308 $\qquad$
i) $\pi$ $\qquad$
k) $-\sqrt{11}$
$\qquad$
h) $\quad 6 . \overline{47}$
f) $\frac{8}{0}$ $\qquad$
j) $\frac{4}{-5}$ $\qquad$
b) 0
d) -5

1) $\frac{0}{-4}$ $\qquad$

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## Answers to each Exercise

## Section 1.1

## Exercise 1:

a) -6
b) +9
c) +1
d) -12
e) - 13
f) +10
g) 3
h) -5
i) is the opposite of
j) is the opposite of
$\mathrm{k})$ is the opposite of

1) is the opposite of

Exercise 2:
a) +12
b) +5
c) +24
d) +1
e) 0
f) +4
a) 3 , to the right
b) 5, to the left
c) value, 7, to the right
d) value, 4, direction, to the left
e) value is 2 and the direction is to the right.

Exercise 4:

Exercise 5:

Exercise 6:

Exercise 7:

Exercise 8:

## Exercise 9:

a).$\overline{5}$
b) $\quad .3 \overline{4}$
c) $\quad 52 . \overline{38}$
d) $-.257918 \overline{40}$
e) $-3.05 \overline{291}$
f).$\overline{21689}$
a) $\frac{9}{10}$
b) $\frac{25}{10}$
c) $\frac{79}{1,000}$
d) $-\frac{3,004}{10,000}$
e) $-\frac{607}{100}$
f) $\frac{9}{100,000}$
a) -5 can be written as a fraction, $-\frac{5}{1}$
b) 2.0815 is a terminating decimal
c) $3.6 \overline{21}$ is a repeating decimal
d) $\frac{6}{-5}$ is a fraction with integers in both the numerator and the denominator.
e) 12 can be written as a fraction, $\frac{12}{1}$
f) $0.89555555 \ldots$ is a repeating decimal
g) $4 \frac{2}{3}$ can be written as a fraction, $\frac{14}{3}$.
a) $\frac{4}{9}$
b) $\frac{6}{9}=\frac{2}{3}$
c) $\quad \frac{26}{99}$
d) $\frac{72}{99}=\frac{8}{11}$
e) $\frac{524}{999}$
f) $\frac{315}{999}=\frac{35}{111}$
a) 7 is natural, whole, an integer and rational
b) 0 is whole, an integer and rational
c) $\frac{-3}{8}$ is rational
d) - 5 is an integer and rational
e) $\sqrt{12}$ irrational
f) $\frac{8}{0}$ is undefined
g) 4.308 is rational
h) $6 . \overline{47}$ is rational
i) $\pi$ is irrational
j) $\frac{4}{-5}$ is rational
k) $-\sqrt{11}$ is irrational
l) $\frac{0}{-4}=0$ so it is whole, an integer and rational

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## Section 1.1 Focus Exercises

1. Find the absolute value of each number, as indicated.
a) $|12|=$
b) $\quad{ }^{0} \mid=$
c) $|-9|=$
d) $\left|-\frac{3}{5}\right|=$
e) $|-2.5|=$
f) $|\sqrt{10}|=$
2. Write each decimal number as a fraction, thereby making it a rational number. Simplify the fraction completely.
a) $.03=$
b) $7.5=$
c) $-0.105=$
3. Write each repeating decimal as a fraction; reduce the fraction if possible.
a) $\quad .8888888 \ldots$
b).$\overline{45}$
c).$\overline{216}$
d) $.72727272 \ldots$
e).$\overline{06}$
f) $.540540540 \ldots$
4. Decide whether each number is rational or not. State why it is or is not.
a) $-\frac{4}{5}$
b) $\frac{0}{4}$
c) $2.031031031031 \ldots$.
d) $5.1929394959 \ldots$
e) 0.590143776
5. Decide into which category or set of the real numbers each fits. Put a check mark in the box(es) that describe that number. Check all that apply.

|  | Natural | Whole | Integer | Rational | Irrational | Undefined |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| a |  |  |  |  |  |  |

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