Section 1.8 The Definitions of Algebra

INTRODUCTION

Algebra is all about symbols and abbreviations, using them and manipulating them. In fact, we use symbols for the six basic operations,

Sum:	addition (+)	as in the sum of 3 *and* 7.
Difference:	subtraction (–)	as in the difference of 9 *and* 2.
Product:	multiplication (⋅)	as in the product of 4 *and* 6.
Quotient:	division (÷)	as in the quotient of 20 *and* 5.
Power:	the exponent	as in the fourth power of 3.
Square root:	the radical	as in the square root of 64.

Notice that the four basic operations all include the word *and* in their written form, but *power* and *square root* do not.

The Greek mathematician Diophantus was one of the first to use symbols to *abbreviate* mathematical expressions and thought. Though his symbols were different from what we use today, he is credited with taking the discussion of mathematics and problem solving from sentence form into symbolic form. Could you imagine having to *discuss* everyday mathematics without the use of a plus sign or a times sign?

Letting symbols represent mathematical ideas allows us to *abbreviate* those ideas. A mathematical sentence written in words (English, not Greek) can be more easily understood if it is written symbolically, as long as you know what the symbols mean.

For example, the simple English expression "the sum of ten and six" can be easily abbreviated as the numerical expression $10 + 6$.

Example 1: Abbreviate each expression with familiar symbols:

	Expression	**Abbreviation**
a)	The sum of twelve and eight:	$12 + 8$
b)	The product of five and nine:	$5 \cdot 9$
c)	The difference of eight and three:	$8 - 3$
d)	The quotient of fourteen and two:	$14 \div 2$
e)	The sixth power of five:	5^6
f)	The square root of 16:	$\sqrt{16}$

Exercise 1: Abbreviate each expression with familiar symbols. You do not need to evaluate these.

a) The product of seven and nineteen.

b) The difference of twenty and eleven.

c) The fourth power of six.

d) The quotient of fifteen and three.

e) The sum of eight and twelve.

f) The square root of nine.

Example 2: Write each abbreviation in English. Note that the four basic operations all require the word *and* in their translation, but *power* and *square root* do not.

	Abbreviation	**Expression**
a)	$7 + 19$:	The sum of 7 and 19.
b)	$24 \div 3$:	The quotient of 24 and 3.
c)	$8 \cdot 12$:	The product of 8 and 12.
d)	$15 - 6$:	The difference of 15 and 6.
e)	3^8:	The eighth power of 3.
f)	$\sqrt{36}$:	The square root of 36.

Exercise 2: Write each abbreviation in English.

a) $24 \div 6$

b) $25 - 9$

c) 5^7

d) $3 \cdot 16$

e) $14 + 20$

f) $\sqrt{49}$

We also use symbols to represent abbreviations and unknown numbers. Some of these abbreviations are:

1. **multiplication**: an abbreviation for repeated addition

   for example, $3 \cdot 5 = 5 + 5 + 5 = 15$

2. the **exponent**: an abbreviation for repeated factors

   for example, $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

3. the **radical**: an abbreviation for the square root

   for example, $\sqrt{49} = 7$

Think about it: The four basic operations all require the word *and* in their translation, but *power* and *square root* do not. Why is that?

____

____

Answer: The four basic operations all require *two* numbers on which to operate, whereas both *power* and *square root* operate on just *one* number.

Back to Top

SOME DEFINITIONS OF ALGEBRA

A **variable** is a letter that represents a number. Sometimes we know what number the variable represents, but usually we don't know. The word "variable" means "vary -able," or "able to vary," "able to change."

In particular, a variable is able to change its value depending on the situation. Sometimes, for example, the letter *w* might have the value **5** and another time it might have the value **7**.

A variable, like an operation sign and the radical, is also a symbol that is a part of the pieces that make up algebra. This leads us to the definition of an *algebraic expression*.

> An **algebraic expression** is, at its greatest, a series of numbers and letters connected by the operations and grouping symbols with which we are most familiar.
>
> At its simplest, an algebraic expression can be just a single variable, such as *y*, or even a single number, such as *3*.

Technically, algebraic expressions that include only numbers and operations—but no variables—are called *numerical expressions*.

Here are some examples of **algebraic expressions**:

$w + 8$ $\quad$ $5 \cdot x + 6 \cdot y^2$ $\quad$ $\sqrt{3 \cdot y - \frac{x}{4}}$ $\quad$ $\frac{2 \cdot x^2 \cdot y + 4 \cdot w}{1 - 5 \cdot z}$

In algebraic expressions, when using variables, we have a slightly simpler way to represent multiplication. If there is no other operation between a number and a variable, or between two variables, then the operation is (automatically) assumed to be multiplication.

For example, $5 \cdot x$ can be written as just 5x. This can also be written as 5(x).

Likewise, xy means $x \cdot y$ This can also be written as (x)(y).

and $6 \cdot x \cdot y^2$ can be abbreviated as $6xy^2$. This can also be written as $6(x)(y)^2$.

Of course, if we wish to represent the product between two numbers, then we must clearly show the multiplication. For example, $5 \cdot 3 = 15$ should never be written as 53. (53 is, of course, fifty-three, and should not be treated as a product.) It could, though, be written as (5)(3) = 15.

In an algebraic expression, numbers are called **constants** because their value doesn't ever change. For example, in the expression x + 3, x is a variable and 3 is a constant.

We do, however, have a special name for the numbers that multiply variables: they are called *coefficients*. A **coefficient** is a number that is a multiplier of a variable. For example, in 7x, 7 is the coefficient of x.

As you know, the commutative property of multiplication allows us to write a product in a different order. For example, $5 \cdot y$ can also be written as $y \cdot 5$. Though we might write $5 \cdot y$ as just **5y**, it's unusual to write $y \cdot 5$ as **y5**. In other words, when writing the product of a number (coefficient) and a variable, we always write the coefficient first.

Back to Top

VARIABLES AS NUMBERS

All variables represent numbers, and that's all. Sometimes we know what number a variable represents, and sometimes we don't. When we are given the numerical value of a variable, that number is called a **replacement value**. That's because we *replace* the variable with the number (the value). We can also say that we **substitute** the value for the variable.

If the replacement value is a negative number, then we should use parentheses around the replacement value.

Example 3: Substitute the variable with the replacement value given, then evaluate the result.

a) $w + 5$ replace **w** with 8: $8 + 5 = 13$

b) $7y$ replace **y** with 4: $7 \cdot 4 = 28$

c) $12 \div x$ replace **x** with - 3: $12 \div (-3) = -4$

d) $9 - p$ replace **p** with - 2: $9 - (-2) = 9 + 2 = 11$

e) c^2 replace **c** with - 5: $(-5)^2 = (-5)(-5) = +25$

Exercise 3: Substitute the variable with the replacement value given, then evaluate the result.

a) $6 + y$ replace **y** with 15:

b) $-8 - w$ replace **w** with - 6:

c) $3x$ replace **x** with 12:

d) $n \div 4$ replace **n** with - 32:

e) v^2 replace **v** with - 7:

It's also possible to replace, or substitute, a variable with more than one value, but the each new value must be treated separately. For example, in the **expression** $2W + 2L$, we see two variables, W and L. They can each have their own replacement value, and they don't need to be the same. For example, in the expression $2W + 2L$, W could be 5 and L could be 8.

Example 4: Substitute the variables with the replacement values given, then evaluate the result.

a) $w + y$ Replace **w** with 8 and **y** with 9: $8 + 9 = 17$

b) $2xy$ Replace **x** with 5 and **y** with - 6: $2 \cdot 5 \cdot (-6) = -60$
(Remember, 2xy means $2 \cdot x \cdot y$.)

c) $a^2 - c$ Replace **a** with 3 and **c** with - 7: $3^2 - (-7) = 9 + 7 = 16$

d) $2W + 2L$ Replace **W** with 5 and **L** with 8: $2 \cdot 5 + 2 \cdot 8 = 10 + 16 = 26$

e) $\dfrac{p + 2}{q \div 3}$ Replace **p** with 8 and **q** with 15: $\dfrac{8 + 2}{15 \div 3} = \dfrac{10}{5} = 2$

Exercise 4: Substitute the variables with the replacement values given, then evaluate the result.

a) $x - p$ replace **x** with - 6 and **p** with 7: ______________________

b) $2wc$ replace **w** with - 5 and **c** with 19: ______________________

c) $\frac{1}{2} \cdot h \cdot b$ replace **h** with 12 and **b** with 9: ______________________

d) $(r + n)^2$ replace **r** with 2 and **n** with 6: ______________________

e) $\frac{y + 4}{3k}$ replace **y** with 26 and **k** with - 2: ______________________

f) $c^3 - y$ replace **c** with 2 and **y** with - 10: ______________________

Sometimes a variable, say w, will appear in more than part of an expression. When this happens, the replacement value for w will be the same for each occurrence of w.

For example, in the **expression** $3 \cdot w + 20 \div w$, we see two w's. If the replacement value for w is 4, then each w has the value of 4. It is not possible for one w to be 4 and the other w to be, say, 7. If you want the variables to have two different values at the same time, then you the expression really needs to have two different variables.

Example 5: Substitute the variable with the replacement value given, then evaluate the result.

a) $2w + 20 \div w$ replace **w** with 4: $2 \cdot 4 + 20 \div 4 = 8 + 5 = 13$

b) $x^2 + 3x - 7$ replace **x** with - 5: $(-5)^2 + 3 \cdot (-5) - 7 = 25 + (-15) - 7 = 3$

c) $4x + xy$ replace **x** with 3 and **y** with - 2: $4 \cdot 3 + 3 \cdot (-2) = 12 + (-6) = 6$

Exercise 5: Substitute the variable with the replacement value given, then evaluate the result.

a) $3x + 7x$ replace **x** with 5:

b) $y^2 + 5y$ replace **y** with - 6:

c) $w \cdot v + 5w$ replace **w** with - 4 and **v** with 9:

d) $P + Pr$ replace **P** with 100 and **r** with .06:

Back to Top

TRANSLATING BETWEEN ENGLISH AND ALGEBRA

As has been stated, variables represent numbers, especially numbers whose values we do not yet know. For example, if we need to represent the sum of *a number* and 5, we can write this as $x + 5$. We recognize $x + 5$ as a sum, and we have used the x to represent *a number*.

Why would we need to consider writing expressions such as $x + 5$? A simple example is presented here. It may not be very exciting, but it shows the use of algebra in somewhat real terms:

> Scott just turned 25 when his daughter Jennifer was born. In fact, Jennifer was born on Scott's 25th birthday. As Jennifer grew older, she began to understand how to calculate her dad's age. She realized that Dad's age (she didn't call him "Scott") was the sum of her age and 25.
>
> To think about how old her dad might be at various stages in her life, Jennifer represented her own age as J and was able to write an expression for her dad's age as $J + 25$. Then she thought,
>
> "When I'm 18 and graduate from high school, Dad will be $18 + 25 = 43$; when I'm 22 and graduate from college, Dad will be $22 + 25 = 47$; when I'm 30 and start a family of my own, Dad will be $30 + 25 = 55$;" a good age to be a grandpa.

In general, if we need to express an unknown number, then we can use any variable we choose. My favorite variable is x, and I use it for most of the unknown numbers that I encounter. Sometimes students ask me to figure out what score they need to get on the final exam in order to get an A or a B in the class. There is a formula that I show them and use x as their unknown final exam score. They can then solve the formula (as you'll be able to do in Chapter 2) and know what score they need get to meet their goals.

Example 6: Translate each English expression into an Algebraic expression. Use any variable of your liking to represent the unknown number (I'll use x).

a) The sum of a number and 18: $x + 18$

b) The difference of a number and 3: $x - 3$

c) The difference of 17 and a number: $17 - x$

d) The product of a number and 6: $x \cdot 6$ or $6x$ (but *not* $x6$)

e) The quotient of a number and 5: $x \div 5$

f) The quotient of 20 and a number: $20 \div x$

g) The square of a number: x^2

h) The square root of a number: $\sqrt{x}$

Exercise 6: Translate each English expression into an Algebraic expression. Use any variable of your liking to represent the unknown number.

a) The product of 5 and a number:

b) The quotient of a number and 4:

c) The difference of 6 and a number:

d) The sum of 11 and a number:

e) The square of a number:

f) The difference of a number and 9:

g) The square root of a number:

Back to Top

TRANSLATIONS AND THE MAIN OPERATION

Sometimes an algebraic phrase or expression has more than one operation in it. When this happens in English it *could* be a little confusing as to how the operations should be written in algebra. There is, however, one key element within the expression that allows us to be very consistent in our translation from English to Algebra, and that is the notion of the *main operation*. Recall that the **main operation** (from Section 1.6) is that operation that is applied *last* according to the order of operations.

For example, the algebraic expression $2x + 5$ has both multiplication (a product) and addition (a sum). Would it be appropriate to call $2x + 5$ a product or a sum?

Since addition has a lower rank than multiplication, it is applied last; hence, addition is the *main* operation. This means that the expression $2x + 5$ is a sum, referring to its main operation:

$2x + 5$ is the sum of the terms $2x$ and 5.

If we had a slightly different expression, say $2(x + 5)$, then we still have both multiplication and addition in it. However, this time addition has a higher rank because it is in the grouping symbols and must be applied first.

Therefore, multiplication is the main operation because it is applied last. This means that the expression $2(x + 5)$ is a product:

$2(x + 5)$ is the product of the factors 2 and $x + 5$.

Example 7: Identify the expression by its main operation, then write (in English) its meaning.

	Expression	Main operation	in English
a)	$x^2 - 3$	subtraction	the difference of x^2 and 3
b)	$\sqrt{x + 2}$	radical	the square root of $x + 2$
c)	$\sqrt{x} + 2$	addition	the sum of $\sqrt{x}$ and 2
d)	$(20 \div x)^4$	exponent	the fourth power of $20 \div x$
e)	$20 \div x^4$	division	the quotient of 20 and x^4

Exercise 7: Identify the expression by its main operation, then write (in English) its meaning.

	Expression	Main operation	in English
a)	$10 + x^2$	______________	____
b)	$\sqrt{25 - x}$	______________	____
c)	$x^2 \div 4$	______________	____
d)	$6(4 - x)$	______________	____
e)	$(x + 1)^5$	______________	____
f)	$\sqrt{x} - 8$	______________	____

Each of the expressions in Example 7 and Exercise 7 have two operations, the main operation and a "sub-operation." For example, in the expression $4x - 3$, the *main* operation is *subtraction*, so this expression is a *difference*. However, within this expression is a *sub-operation*, multiplication, as in the product of 4 and x, or as the product of 4 and a number. We might even call this a "sub-expression" within the larger expression.

In translating to an English form, we could show any sub-expression in parentheses.

For example, once we recognize the main operation of $4x - 3$ as subtraction, we can write that it is "the difference of (4x) and 3." The purpose of this is to recognize another translation within the expression.

By itself, $4x$ is "the product of 4 and a number," so putting the whole expression into words, $4x - 3$ becomes

"The difference of the product of 4 *and* a number *and* 3."

Notice that
- the "sub-expression" has been underlined. This helps us distinguish between the two times that the word 'and' appears in the expression;
- the *main* operation is written *first*, difference;
- the sub-expression always begins with the word representing the sub- operation;
- the four basic operations requires the word 'and', but operations like "square," "power" and "square root" do not.

Let's put this into practice.

Example 8: In each English expression, put a [box] around the main operation (always written first) and the [and] to which it applies (if any); also, underline the whole sub-expression. Write the expression's meaning in algebra. (If two 'ands' appear, then one is for the main operation and the other is for the sub-operation.)

	English Expression	**in Algebra**
a)	The [sum] of the product of 2 and a number [and] 5.	$2x + 5$
b)	the [difference] of 2 [and] the sum of a number and 5.	$2 - (x + 5)$
c)	the [product] of 3 [and] the square of a number.	$3x^2$
d)	the [square root] of the sum of a number and 2.	$\sqrt{x + 2}$
e)	the [sum] of the square root of a number [and] 2.	$\sqrt{x} + 2$
f)	the [fourth power] of the quotient of 20 and a number.	$(20 \div x)^4$
g)	the [quotient] of 20 [and] the difference of a number and 8.	$20 \div (x - 8)$

Exercise 8: In each English expression, put a box around the main operation (always written first) and the and to which it applies (if any); also, underline the sub-expression, then write its meaning in algebra.

English Expression	in Algebra
a) The quotient of the sum of 5 and a number and 6.	________________
b) The product of 6 and the difference of 4 and a number.	________________
c) The sum of 8 and the product of 5 and a number.	________________
d) The difference of 10 and the product of 9 and a number.	________________
e) The sum of 10 and the square of a number.	________________
f) The square root of the product of 25 and a number.	________________
g) The quotient of the square of a number and 4.	________________
h) The fifth power of the sum of a number and 1.	________________
i) The difference of the square root of a number and 8.	________________
j) The difference of 9 and the sum of a number and 4.	________________

Answers to each Exercise

Section 1.8

Exercise 1:

a) $7 \cdot 19$ b) $20 - 11$ c) 6^4

d) $15 \div 3$ e) $8 + 12$ f) $\sqrt{9}$

Exercise 2:

a) the quotient of 24 and 6 b) the difference of 25 and 9

c) the seventh power of 5. d) the product of 3 and 16

d) the sum of 14 and 20 f) the square root of 49.

Exercise 3:

a) $6 + 15 = 21$ b) $-8 - (-6) = -2$ c) $3 \cdot 12 = 36$

d) $-32 \div 4 = -8$ e) $(-7)^2 = 49$

Exercise 4:

a) $(-6) - 7 = -13$ b) $2 \cdot (-5) \cdot 19 = -190$

c) $\frac{1}{2} \cdot 12 \cdot 9 = 6 \cdot 9 = 54$ d) $(2 + 6)^2 = (8)^2 = 64$

e) $\frac{26 + 4}{3 \cdot (-2)} = \frac{30}{-6} = -5$ f) $2^3 - (-10) = 8 + 10 = 18$

Exercise 5:

a) $3 \cdot 5 + 7 \cdot 5 = 15 + 35 = 50$ b) $(-6)^2 + 5 \cdot (-6) = 36 + (-30) = 6$

c) $(-4) \cdot 9 + 5 \cdot (-4) = -36 + (-20) = -56$

d) $100 + 100 \cdot (.06) = 100 + 6 = 106$

Exercise 6:

a) $5 \cdot x$ or $5x$ b) $x \div 4$ c) $6 - x$ d) $11 + x$

e) x^2 f) $x - 9$ g) $\sqrt{x}$

Exercise 7:

a) addition; the sum of 10 and x^2

b) radical; the square root of $25 - x$

c) division; the quotient of x^2 and 4

d) multiplication; the product of 6 and $(4 - x)$

e) exponent; the fifth power of $(x + 1)$

f) subtraction; the difference of $\sqrt{x}$ and 8

Exercise 8:

a) $(5 + x) \div 6$ b) $6 \cdot (4 - x)$ c) $8 + 5 \cdot x$

d) $10 - 9 \cdot x$ e) $10 + x^2$ f) $\sqrt{25 \cdot x}$

g) $x^2 \div 4$ h) $(x + 1)^5$ i) $\sqrt{x} - 8$

j) $9 - (x + 4)$

Back to Top

Section 1.8 Focus Exercises

1. Abbreviate each expression with familiar symbols. You do not need to evaluate these.

a) The difference of fourteen and two. ________________

b) The square root of sixteen. ________________

c) The sum of fifteen and ten. ________________

d) The third power of five. ________________

e) The product of eleven and eight. ________________

f) The quotient of twelve and four. ________________

2. Write each abbreviation in English.

a) $9 \cdot 20$ ________________________________

b) $\sqrt{64}$ ________________________________

c) $16 - 4$ ________________________________

d) $18 \div 3$ ________________________________

e) 3^5 ________________________________

f) $8 + 10$ ________________________________

3. Translate each English expression into an Algebraic expression. Use any variable of your liking to represent the unknown number.

a) The product of -3 and a number: ____________________

b) The quotient of a number and 7: ____________________

c) The difference of -9 and a number: ____________________

d) The sum of - 10 and a number: ____________________

e) The square of a number: ____________________

f) The difference of a number and - 8; ____________________

4. Substitute the variable with the replacement value given, then evaluate the result.

a) $y - k$ replace **y** with - 4 and **k** with (- 2): ____________________

b) $-3mh$ replace **m** with - 5 and **h** with - 6: ____________________

c) $\frac{y + 6}{-2k}$ replace **y** with - 42 and **k** with - 2: __________

d) $b^2 + x$ replace **b** with - 3 and **x** with - 10: ____________________

e) $y^2 - 5y$ replace **y** with - 2: ____________________

f) $w \cdot v + 6w$ replace **w** with - 3 and **v** with 5: ____________________

5. Identify the expression by its main operation, then write (in English) its meaning.

	Expression	Main operation	in English
a)	$x^2 + 6$	__________	______________________________
b)	$\sqrt{x - 5}$	__________	______________________________
c)	$(3 \cdot x)^2$	__________	______________________________
d)	$6(x + 8)$	__________	______________________________
e)	$10 - \sqrt{x}$	__________	______________________________

6. In each English expression, put a box around the main operation (always written first) and the and to which it applies (if any); also, underline the sub-expression, then write its meaning in algebra.

English Expression		in Algebra
a)	The quotient of the square of a number and 4.	______________________
b)	The product of 6 and the sum of a number and 2.	______________________
c)	The sum of 3 and the quotient of a number and 6.	______________________
d)	The difference of 5 and the product of a number and 9.	______________________
e)	The difference of the square root of a number and 9.	______________________
f)	The square of the sum of a number and 6.	______________________

Back to Top