1.9 Algebraic Expressions

TERMS

In algebra, when a constant is multiplied by a variable, like $5 \cdot y$, the constant, *5*, is called a **coefficient**, and the product of a coefficient and a variable create a **term**. So in the *term* 5y, the *coefficient* is 5.

We don't need to restrict *terms* to just the product of a coefficient and a variable, a term can be:

1. the product of two or more variables: $a \cdot b$, $w \cdot z$, or $x \cdot y$ (or just ab, wz, xy);
2. the product of the same variable with itself—any number of times: $x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$;
3. the product of a constant (coefficient) with a variety of variables: $8 \cdot a^5 \cdot b^3$, or $8a^5b^3$;
4. just a constant with no variable: 6, -5, or $\frac{3}{8}$.

This leads to two new definitions:

1. A **variable cluster** can be just one variable *or* can be the product of more than one variable, *each with its own exponent*. A **variable cluster** does *not* include the coefficient.

2. A **term** can be just a constant *or* can be the product of a constant—called a coefficient—and a variable cluster.

Example 1: Given the term identify the coefficient and the variable cluster.

Term	Coefficient	Variable Cluster	Term	Coefficient	Variable Cluster
$8x^2y^3$	8	x^2y^3	$5x^4$	5	x^4
3a	3	a	$-6x^2$	-6	x^2
$\frac{2}{3}cb^5$	$\frac{2}{3}$	cb^5	7	7 *	(none)

* 7 is a term that is really a *constant* term; it is not, itself, a coefficient as defined above.

Here's a special term: **x**. "What's so special?" you ask? Well, it *appears* that this term has no coefficient (nor any exponent). Actually, it has both. In fact,

- x is the same as 1x or $1 \cdot x$;
- it's also the same as x^1 (read "x to the first power");
- it could even be thought of as $1x^1$.

This may look as if we're getting carried away, but the 1's that you see here are actually very important numbers and deserve more than a passive recognition. In both cases, they are sometimes referred to as, "the invisible 1." Though it may sound a little like the twilight zone, the "invisible 1" will rush to our aid—and make itself visible—in many situations.

Example 2: Given the term identify the coefficient and the variable cluster.

Term	Coefficient	Variable Cluster	Term	Coefficient	Variable Cluster
x^2y^3	1	x^2y^3	$-x^4$	- 1	x^4
- a	- 1	a	x^2	1	x^2

Exercise 1: Given the term identify the coefficient and the variable cluster.

	Term	Coefficient	Variable Cluster		Term	Coefficient	Variable Cluster
a)	$3x^5y$			b)	- 4		
c)	- 2c			d)	$-x^2$		
e)	$\frac{7}{8}c^2d^5$			f)	w^8		
g)	$6x^0$			h)	y		
i)	$-n^5$			j)	- y		

To top of page

ALGEBRAIC EXPRESSIONS

You've already been introduced to the notion of algebraic expressions. Here it is again with a little explanation.

In algebra we tend to put things, such as terms, together. We connect terms with parentheses and operations and call them **algebraic expressions**. Here are some examples of algebraic expressions:

$9y + 5x$	the sum of two terms
$3a(4b - 7)$	the product of a term and a quantity
$(8x^2y^3)(2xy)$	the product of two terms
$(5x + 6)(2x - 1)$	the product of two quantities
$7x^2 + 6x + 1$	the sum of three terms
$\frac{4b}{9a}$	the quotient of two terms
$6y^3$	A single term is also an algebraic expression
12	Even a constant could be thought of as an algebraic expression.

At this point, for Example 3, we're going to restrict our discussion of algebraic expressions to individual terms and to the *sum* of two or more terms.

Example 3: Identify the terms in each (algebraic) expression.

	Expression	terms (separated by commas)	
a)	$7x^2 + 6x + 1$	three terms:	$7x^2$, $6x$, 1
b)	$5x + 2$	two terms:	$5x$, 2
c)	$-3a^5 + 2a^3 + 4a + 9$	four terms:	$-3a^5$, $2a^3$, $4a$, 9
d)	$-15y$	one term:	$-15y$

Of course, there are many algebraic expression that have subtraction as the operation within, such as $3x^2 - 8x - 2$. This expression has three terms, but we need to remember a rule from Section 1.2 to know what the terms are:

The sign in front of a number belongs to that number.

For algebraic expressions, we extend that notion to include terms; it now reads

The sign in front of a *term* belongs to that *term*.

So, the terms of $3x^2 - 8x - 2$ are $\mathbf{3x^2}$, $\mathbf{-8x}$ and $\mathbf{-2}$.

Example 4: Identify the terms in each (algebraic) expression.

	Expression	terms (separated by commas)	
a)	$7x^2 - 6x - 1$	three terms:	$+7x^2$, $-6x$ *and* -1
b)	$5x^2 - 2x$	two terms:	$+5x^2$ *and* $-2x$
c)	$3a^5 - 2a^3 - 4a + 9$	four terms:	$+3a^5$, $-2a^3$, $-4a$ *and* $+9$

Exercise 2: Identify the terms in each (algebraic) expression.

	Expression	Terms (separate them by a comma)
a)	$5x^2 - x + 3$	______________
b)	$x^3 - 6x^2 - 8x$	______________
c)	$-2x - 6x^5$	______________
d)	$-3x^2y^2 + 6x^2 + 1x^2 - 4$	______________
e)	$4 - \frac{2}{3}x - y$	______________

To top of page

LIKE TERMS

Two things are "like" each other if they have the same characteristics. For example *like* fractions have the same denominator. In algebra, two or more terms are considered to be **like terms** if they have exactly the same variable cluster, even if the coefficients are different. For example, $3x^2y^5$ and $9x^2y^5$ are like terms because they have *exactly* the same variable cluster, x^2y^5.

> Two or more terms are considered to be **like terms** if they have exactly the same *variable cluster*.

Furthermore, constants, which have no variable cluster, are "like" all other constants. Use the example below to carefully examine which terms are "like" and which are *not* like. See if you can decide why some terms are not like others within the same expression.

Example 5: In each expression, identify the like terms.

	Expression		**Like terms**
a)	$7x - 6x + 1$	one pair of terms is **like**:	$7x$ and $-6x$
b)	$5x - 2 + 9$	one pair of terms is **like**:	-2 and 9
c)	$3a^2 + 2b^3 + 4a^2 + 9b^3$	two pairs of terms are **like**:	$3a^2$ and $4a^2$, $2b^3$ and $9b^3$
d)	$-15y + 6y - 3y$	all three terms are **like**:	$-15y$ and $6y$ and $-3y$
e)	$4x + 6y - 7$	no terms are **like**:	(none)
f)	$5ab - 8ab + 3b$	one pair of terms is **like**:	$5ab$ and $-8ab$

(Note: the variable cluster for **3b** is just **b** which is not the same as **ab**.)

g)	$-2x^3 + 6x^2 + 9x$	no terms are **like**:	(none)

(Note: a different exponent on the same variable indicates a different variable cluster.)

Exercise 3: In each expression, identify the like terms.

	Expression	Like Terms
a)	$5x - x + 3$	______________
b)	$x^2 - 6x - 8x^2$	______________
c)	$2x^5 - 6x^5$	______________
d)	$-3x^2y + 6x^2 + 1x^2y - 4$	______________
e)	$4 - \frac{2}{3}y - y$	______________

To top of page

COMBINING LIKE TERMS

In algebra you will be asked to "simplify" a great deal. Unfortunately, this may cause some confusion at times because "simplify" will have different meanings based on the expression given and under what circumstances the expression is being used.

Basically, an expression is **simplified** when it is written in the briefest, least complex form. When working with like terms, an expression is said to be simplified when the like terms are "combined." To **combine** like terms means to find their sum or difference, depending on the operation between the terms.

We can use *like measurements*, such as <u>inches and inches</u> or <u>feet and feet</u>, as an illustration:

2 inches + 3 inches = 5 inches.

2 feet + 3 feet = 5 feet.

In that analogy we are able to combine the measurements because they are "like measures," meaning the unit of measurement (inches or feet) is the same. Notice that the result, the sum, is in the same unit of measurement, feet, as the addends. In the very same way, we can combine like terms because the variable cluster is the same:

$$2x + 3x = 5x.$$

Again, the result, **5x**, has a variable cluster that is exactly the same as the terms, **2x** and **3x**. This means that the result is "like" the original two terms. In this way, the only change is the coefficient; to get the coefficient of the result, 5, we had to add the coefficients of the like terms, 3 and 2.

Why is it that combining like terms works this way? Good question. It goes back to the abbreviation called "multiplication." Remember that multiplication is just an abbreviation for repeated addition. So, when we wish to find the sum of $4h$ and $3h$, it's like this:

$4h$ means $4 \cdot h$ which means $h + h + h + h$. Likewise, $3h$ means $3 \cdot h$ which means $h + h + h$. Therefore, when we add,

$$4h \quad + \quad 3h$$

is the same as $(h + h + h + h) + (h + h + h)$

or just $h + h + h + h + h + h + h$ which is abbreviated as $7 \cdot h$, or just $7h$.

This is also why we can't combine $2y + 5x$. It becomes

$$(y + y) + (x + x + x + x + x)$$

In this case, we can't say that we have 7 of anything that is the same.

The combining of like terms is not restricted to addition; we can also *subtract* like terms. In this way it is like eliminating terms, maybe one at a time. In elementary school, subtraction is sometimes referred to as "take aways," such as 5 – 2 means "5 take away 2". It's like cutting 2 feet off of a 5 foot board.

5 feet – 2 feet = 3 feet.

Or, five x's take away two x's leaves three x's: $5x - 2x = 3x.$

In this case we *combine* like terms by finding the difference between the coefficients.

> To **combine** like terms we need only add or subtract their coefficients.

However, if terms are not *like* terms, then they cannot be combined. A somewhat silly example might be to add 6 feet and 5 pounds. They have no relevance to each other and cannot combine. Similarly, the expression $6x + 5y$ cannot be rewritten as one term. $6x + 5y$ is simplified just as it is.

Combining like terms is an art that needs to be mastered in order to be successful in algebra. It is not difficult, but you do need to learn it! Let's practice:

Example 6: Simplify each expression by combining like terms. (Find the sum of the coefficients.)

First, written "vertically":

a) $\begin{array}{r} 10x \\ +\ 4x \\ \hline 14x \end{array}$

b) $\begin{array}{r} 12b^4 \\ +\ 7b^4 \\ \hline 19b^4 \end{array}$

c) $\begin{array}{r} 2h \\ 3h \\ +\ 6h \\ \hline 11h \end{array}$

Second, written "horizontally":

d) $7x + 6x = 13x$

e) $-2 + 9 = 7$

f) $-3a^2 + 4a^2 = 1a^2$ or just a^2

g) $4y + 1y = 5y$

h) $4p + p = 5p$

i) $15y^3 + 6y^3 + 3y^3 = 24y^3$

j) $-2x^2y^3 + 9x^2y^3 = 7x^2y^3$

k) $2x^4y^5 + 9x^5y^4$ <u>cannot combine</u>

Notice that (g) and (h) are virtually the same; what we do see in (g) but don't see in (h) is the "invisible 1," the coefficient of **p**. This is a time when making the "invisible 1" *visible* might be helpful to the process, though it isn't necessary.

Also notice that, in (k), $2x^4y^5$ and $9x^5y^4$ are not like terms, so they can't combine. Look carefully at the variable cluster of each term to see why.

Example 7: Simplify each expression by combining like terms. Find the difference of the coefficients.

a) $7w - 2w = 5w$

b) $3x^2 - 2x^2 = 1x^2$ or just x^2 The coefficient 1 doesn't need to be written.

c) $18x - 6x = 12x$

d) $9y - y = 9y - 1y = 8y$ Here, it might be to your advantage to include the coefficient 1 before subtracting.

Of course, since the new rule states that "the sign in front of the term belongs to that term," even subtraction of terms is an addition process. It's just like adding (or subtracting) two signed numbers.

Just as 3 – 8 can be thought of as *+3 and -8* (this adds to - 5),

so can 3x – 8x can be thought of as *+3x and -8x* (this adds to - 5x).

Here's the bottom line on combining like terms:

> To **combine like terms**, simply *add (or subtract) the numerical coefficients*; the result you get will be "like" the original terms. In other words, the variable cluster will remain the same. If a variable cluster has no visible coefficient, then the coefficient is 1.

Example 8: Simplify each expression by combining like terms.

a) $-7w - 2w = -9w$

b) $3x^2 - 5x^2 = -2x^2$

c) $-18x + 6x = -12x$

d) $-9y + y = -9y + 1y = -8y$

e) $-2p + 3p = 1p$ or just p

f) $-c + 7c = -1c + 7c = 6c$

g) $-m - (-6m) = -1m + 6m = 5m$

h) $3x - (-4x) = 3x + 4x = 7x$

i) $-4x + 4x = 0x = 0$

h) $5y^2 + 3y$ cannot combine (unlike terms)

Exercise 4: Simplify each expression by combining like terms. If the terms are *not* like terms, then write "cannot combine."

a) $3y + 9y =$ ____________

b) $7y - 2y =$ ____________

c) $-4w + (-2w) =$ ____________

d) $5w^3 - 9w^3 =$ ____________

e) $-6c^2 + 7c^2 =$ ____________

f) $-c - 8c =$ ____________

g) $p + (-4p) =$ ____________

h) $5p^2 - 6p =$ ____________

i) $-8x^2y + 2x^2y =$ ____________

j) $-4x - (-x) =$ ____________

k) $6m + (-6m) =$ ____________

l) $2s - 4m =$ ____________

m) $-3v + 3v =$ ____________

n) $-v - v =$ ____________

To top of page

THE PRODUCT OF TWO TERMS

The product of two terms is another term. The associative property, the commutative property and the definition of exponents combine to make multiplying two terms a relatively easy process.

For example, we can multiply (3x)(2x) using the properties mentioned above. A word of caution, though: this is behind the scenes work to explain what's happening; it is not intended that students will copy this process. Instead, your process will be much simpler than this.

The explanation:

		$(3x)(2x)$	
can be rewritten as	=	$3 \cdot x \cdot 2 \cdot x$	by the associative property
	=	$3 \cdot 2 \cdot x \cdot x$	by the commutative property
	=	$6 \cdot x^2$	$x \cdot x = x^2$
	=	$6x^2$.	

Of course, we would normally do this in just one step: $(3x)(2x) = 6x^2$. The result is the product of the coefficients and the product of any variable factors present.

Example 9: Multiply and write the product as one term.

a) $(4x)(5x) = 20x^2$

b) $(-3y)(8y) = -24y^2$

c) $(w)(-2w) = -2w^2$

d) $3(6x^2) = 18x^2$

e) $(-4y)(-2) = 8y$

f) $(8w)(-4y) = -32wy$

Exercise 5: Multiply and write the product as one term.

a) $(9c)(4c) =$

b) $-6(5y) =$

c) $(3a)(-12a) =$

d) $(-7x)(-x) =$

e) $5y(-4) =$

f) $(-2b)(6c) =$

g) $-9(-5x^2) =$

h) $x(-3) =$

i) $(4d)(-1) =$

j) $w(-w) =$

k) $-y(y) =$

l) $(-x)(-x) =$

m) $3w(-1) =$

n) $-4y(1) =$

o) $(-2x)(-1) =$

To top of page

Exercise 5:	a)	$36c^2$	b)	$-30y$	c)	$-36a^2$	d)	$7x^2$
	e)	$-20y$	f)	$-12bc$	g)	$45x^2$	h)	$-3x$
	i)	$-4d$	j)	$-w^2$	k)	$-y^2$	l)	x^2
	m)	$-3w$	n)	$-4y$	o)	$2x$		

Exercise 6:	a)	$6(4x - 3) = 24x - 18$	b)	$7(9y - 4) = 63y - 28$
	c)	$(2m - 6)\cdot 5 = 10m - 30$	d)	$4(6 - 5p) = 24 - 20p$
	e)	$9(3 - 3w) = 27 - 27w$	f)	$3(8 - 9v) = 24 - 27v$
	g)	$2(2d - 5k) = 4d - 10k$	h)	$1(3x - 4y) = 3x - 4y$

Exercise 7:	a)	$-5(6x + 3) = -30x - 15$	b)	$-2(8k + 10) = -16k - 20$
	c)	$-2(8m - 1) = -16m + 2$	d)	$-4(y - 7) = -4y + 28$
	e)	$-6(-10 + 4w) = 60 - 24w$	f)	$-3(-9 + 5c) = 27 - 15c$
	g)	$-4(-11 - 5p) = 44 + 20p$	h)	$-7(-4 - 2d) = 28 + 14d$

Exercise 8:	a)	$x(x + 3) = x^2 + 3x$	b)	$y(x - 10) = yx - 10y$ or $xy - 10y$
	c)	$-2a(a + 1) = -2a^2 - 2a$	d)	$-3(y^2 - 7y) = -3y^2 + 21y$
	e)	$w(3w + 4) = 3w^2 + 4w$	f)	$6(5w^2 - 1) = 30w^2 - 6$
	g)	$-4m(6m + 1) = -24m^2 - 4m$	h)	$-1(8m - 5x) = -8m + 5x$
	i)	$3c(3c + 6) = 9c^2 + 18c$	j)	$-k(4k - 5) = -4k^2 + 5k$
	k)	$-5x(-2x + 3) = 10x^2 - 15x$	l)	$-4y(3k - 2) = -12yk + 8y$
	m)	$-6p(-p - 5) = 6p^2 + 30p$	n)	$-4w(w - 1) = -4w^2 + 4w$

To top of page

4. Multiply and write the product as one term.

a) $(7k)(6k) =$ b) $-3(8c) =$ c) $(2p)(-14p) =$

d) $(-9x)(-x) =$ e) $8c(-6) =$ f) $(-4m)(3k) =$

g) $-7(-8x^4) =$ h) $x(-2) =$ i) $(6n)(-5) =$

j) $y(-y) =$ k) $-c(c) =$ l) $(-x)(-x) =$

m) $2y(-5) =$ n) $-6c(5) =$ o) $(-4x)(-5) =$

5. Apply the distributive property to each expression.

a) $5(7x - 8)$ b) $8(1 - 2v)$

c) $(9m - 5)6$ d) $3(8x - 7y)$

e) $-6(5x + 8)$ f) $-7(y - 4)$

g) $-5(-3 + 7w)$ h) $-2(-8p - 9d)$

i) $x(x + 4)$ j) $2y(3y - 10)$

k) $-3a(a + 1)$ l) $-4y(y^3 - 7y)$

m) $-5m(7m + 1)$ n) $-1(9m - 6x)$

o) $-k(-5k - 6)$ p) $-7p(-p + 3)$

To top of page