# Section 2.5 Linear Inequalities

## WORDS OF COMPARISON

Recently, you worked with applications (word problems) in which you were required to write and solve an equation. Sometimes you needed to translate sentences from English into algebra. You might have done so by noticing some key words that have certain mathematical meanings and symbols.

For example, the word "twice" is interpreted as "two times," and "3 more than" means " add 3." An especially important math word in an English sentence is the word "is;" it *usually* means "equals," but not always. What the word "is" does mean, though, is a comparison between two expressions or values.

You all know what an equal sign is; it looks like this: **= .** We use it in math to show that two expressions are *equivalent* some how, as in $3x - 5 = 2x + 7$. This **equation** is a **mathematical sentence**.

As an English sentence, $3x - 5 = 2x + 7$ might have come from,

"5 inches less than 3 feet *is the same as* 7 inches more than two feet."

Or, maybe, "3 feet minus 5 inches *is the same as* two feet plus 7 inches."

Within the English sentence, "is the same as," is a comparison of *equality*, and we can use the equal sign (=) to express that. The **equal** sign represents a comparison between two expressions, namely, that they have **the same value**.

However, what if two expressions do not have the same value? What if they are not equal to each other? What if one expression is greater than the other? Or, what if one is less than the other? In each of these cases, we would not have equality; instead we'd have *in*-equality (*not* equal to).

**Inequalities** come in three basic forms:

a) "**is not equal to**" or "**is not the same as**" ( $\neq$ ) as in, "Meryl Streep is not the same age as Tom Cruise." In this example, we could say that Meryl's age is not equal to Tom's age, or

Meryl's age $\neq$ Tom's age.

b) "**is greater than**" ( $>$ ) as in, "The number of people living in San Diego is greater than the number of people living in Riverside," or

San Diego's population $>$ Riverside's population.

c) "**is less than**" ( $<$ ) as in, "The number of points (runs) in a baseball game is less than the number of points in a basketball game," or

baseball game's score $<$ basketball game's score.

In the examples, above, we know each of them to be true. Sometimes, though, we are given a sentence (that compares two things) that we're not sure is always true. For example, "The number of boys is less than the number of girls," may or may not be true. It all depends on the group of kids to which we are referring.

In algebra, it is appropriate to let a variable represent an unknown number. For example, in the sentence, “The number of kids is less than 10,” we can let x = the number of kids and then represent the sentence in algebra as

$$x < 10.$$

This means that x (the number of kids) can be 9 or fewer. In fact, according to the sentence, it’s even possible that there are 0 kids (no kids at all). But, does the sentence allow for the possibility of there being 10 kids? No, not the way the sentence is written.

What if we wanted to include the possibility of 10 kids? We might say that there could be 10 or fewer kids. In mathematics, we could say, “The number of kids is less than or equal to 10.” This kind of sentence allows for two additional forms of inequalities:

d) “**is greater than or equal to**” ( $\geq$ ) as in, “The number of people living in Moreno Valley is greater than or equal to the number of people living in Temecula,” or

Moreno Valley’s population $\geq$ Temecula’s population.

e) “**is less than or equal to**” ( $\leq$ ) as in a soccer game, if the visiting team didn’t win. (In soccer, tie games are allowed; so if the visiting team didn’t win, then they either lost or tied the game.) “The number of points scored by the visiting team is less than or equal to the number of points scored by the home team,” or

the visiting team’s score $\leq$ the home team’s score.

There are special phrases that mean one inequality or another. To translate English into algebra, think about and learn these translations. Here’s the situation. You are preparing for a birthday party for some kids. Think of it as, “How many chairs must be provided in the following situations?”

For each, **Let x = the number of kids going to the party**.

| Phrase | Meaning | Example | Algebra |
|---|---|---|---|
| fewer than | less than | There are **fewer than** 10 kids.<br>(9 chairs would be enough, maybe even fewer than that.) | $x < 10$ |
| more than | greater than | There are **more than** 10 kids.<br>(11 chairs is the minimum needed; 10 chairs would not be enough.) | $x > 10$ |
| at least | greater than or equal to | There are **at least** 10 kids.<br>(10 is the fewest number of chairs needed, but you may need *more* than that.) | $x \geq 10$ |
| at most | less than or equal to | There are **at most** 10 kids.<br>(10 is the most chairs needed, but it could be that you need even fewer than that.) | $x \leq 10$ |
| no fewer than | greater than or equal to | There are **no fewer than** 10 kids.<br>(You can’t have fewer than 10 chairs; you must have *at least* 10—10 or more—chairs.) | $x \geq 10$ |
| no more than | less than or equal to | There are **no more than** 10 kids.<br>(There’s not going to be 11 kids; you might need as many as 10 chairs, but possibly less.) | $x \leq 10$ |

**Example 1:** Translate each of these sentences into algebra. For each, Let x = the number of games.

a) Nikki's team will play more than 8 games this season: $\mathbf{x > 8}$

b) Juan's team will play at least 12 games this season. $\mathbf{x \geq 12}$

c) Lonny's team will play less than 10 games: $\mathbf{x < 10}$

d) Shari's team will play no more than 9 games: $\mathbf{x \leq 9}$

e) Kyle's team will play at most 15 games: $\mathbf{x \leq 15}$

**Exercise 1:** Translate each of these sentences into algebra. For each, Let x = the number of kids.

a) Nate invited less than 15 kids to his birthday party. ___________

b) Sarah has no more than 23 kids in her class. ___________

c) There were at least 20 kids in the play. ___________

d) The basketball league will have at most 8 kids on a team. ___________

e) Mr. Simpson has said that more than 5 kids will get an A. ___________

f) Fewer than 8 kids signed up to go on the field trip. ___________

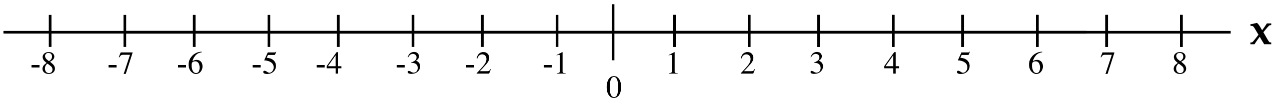
## THE NUMBER LINE

You were introduced to the number line in Chapter 1 and used it to assist you in the addition of signed numbers (positive and negative numbers). Let's look more closely at the number line, especially as it relates to inequalities.

First, you know that variables represent numerical values. In formulas, variables can be just about any number we want them to be. In linear equations, though, we solved and found that the variable was usually equal to just one value.

For this section, we are going to think of variables as "replacement values" (as we did in Section 2.1). This means that the variable x, for example, has the possibility of being any value we choose, either positive, negative, zero, integer, fraction or irrational number. In effect, x can be any *real* number.

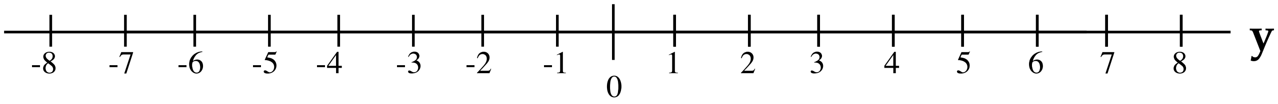
Let’s represent the idea that x can be any real number by placing an ‘x’ to the right of the number line. This will indicate that we are working with values of x and not another variable. We can say that this number line is “in terms of x,” or is “in x.”



This **x** indicates that the number line values are possible values of *x*.

Of course, we can use other variables. Below is a number line “in y.”

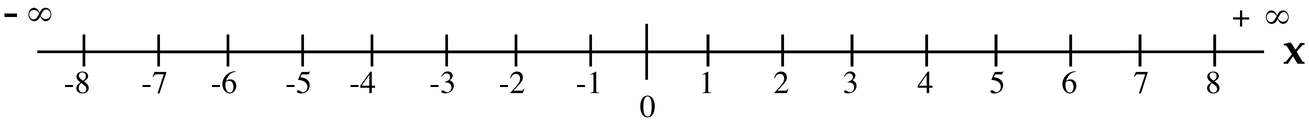
This **y** indicates that the number line values are possible values of *y* .



Next, the number line is not restricted to just the integers shown above. Between each integer is an *infinite* set of fractions and irrational numbers. So, too, are there an infinite set of integers that extend beyond 8 and -8. In other words, the number line goes *infinitely* (indefinitely) to the right (positive) and to the left (negative). There is no “largest” or “greatest” positive number. We sometimes say that the number line goes to **infinity**.

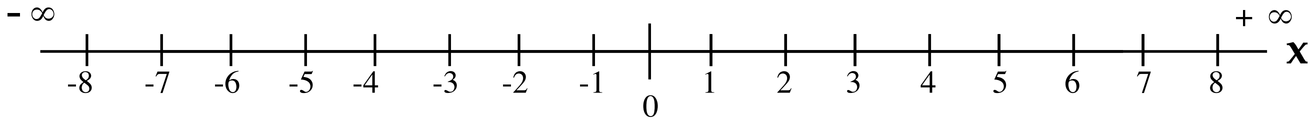
It’s important to understand that **infinity** is not an actual number, it’s just the idea of being “larger than all get out.” The symbol we use for infinity is a “lazy eight”: $\infty$. We use $+\infty$ to indicate numbers that are indefinitely *positively* large, and we use $-\infty$ for those numbers that are indefinitely *negatively* large.

We can even include both positive and negative infinity on the number line. They help remind us that the graph actually extends indefinitely to the left and to the right.

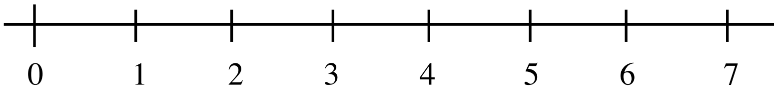


## VISUALIZING INEQUALITIES

We have a way to visualize inequalities, and it has everything to do with the number line. As you know, we place zero (0) in the middle, positive numbers are to the right of zero and negative numbers are to the left.



Let's look, though, at a number line that includes only positive numbers and 0.

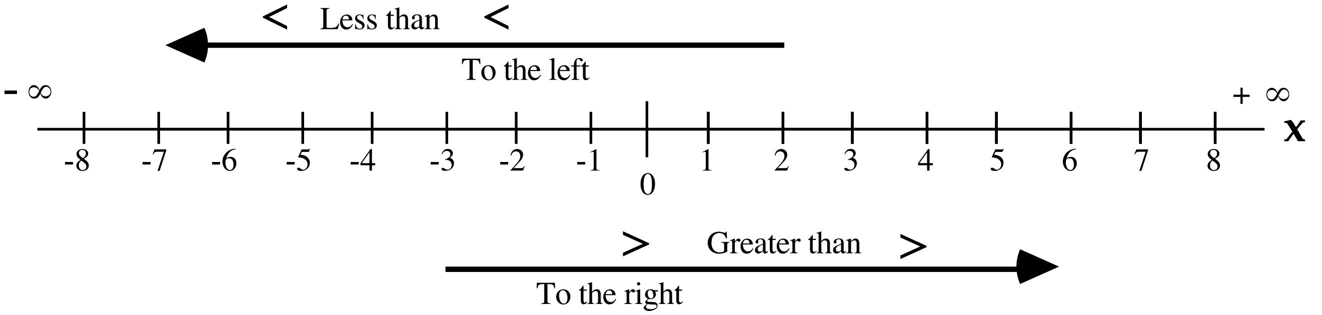


Notice that greater numbers are to the right and lesser numbers are to the left. If we extend this "left-right" notion to the entire number line, then still, lesser numbers are to the left.

That means that <u>negative numbers are less than positive numbers</u>.

It also means that *larger* negative numbers, like -20, are actually, technically, "less than" *smaller* negative numbers, like -5. This is because -20 is *to the left of* -5.

In the end, when we think of numbers along the number line, "less than" means "to the left" and "greater than" means "to the right."



Do you notice anything peculiar about the inequality signs and the direction they point? Notice that the *less than* sign points left and the *greater than* sign points right. This may be helpful when first learning about these new symbols. After a while, as you use them more and more, they'll become easier to distinguish.

In this way, we also talk about the **direction** of the inequality sign; *less than* < "points" to the left and *greater than* > points to the right.

**Example 2:** Fill in the box with an inequality sign (either < or >) that makes the statement true. (You may use the number line, below, to help you think about the answers.)

a) $2 \ \square \ 9$

b) $8 \ \square \ 6$

c) $3 \ \square \ -5$

d) $-6 \ \square \ 1$

e) $-20 \ \square \ -5$

f) $-4 \ \square \ -9$

**Answer:** Use the number line as a guide and the idea that *less than* means *to the left* and that *greater than* means *to the right.*

a) $2 \boxed{<} 9$

2 is to the left of 9

b) $8 \boxed{>} 6$

8 is to the right of 6

c) $3 \boxed{>} -5$

3 is to the right of - 5

d) $-6 \boxed{<} 1$

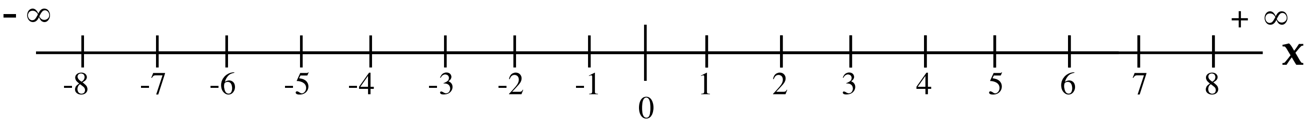
- 6 is to the left of 1

e) $-20 \boxed{<} -5$

- 20 is to the left of - 5

f) $-4 \boxed{>} -9$

- 4 is to the right of - 9



**Exercise 2:** Fill in the box with an inequality sign (either < or >) that makes the statement true. (You may use the number line, above, to help you think about the answers.)

a) $8 \ \square \ 3$

b) $7 \ \square \ -8$

c) $-6 \ \square \ 2$

d) $-4 \ \square \ 9$

e) $-14 \ \square \ -3$

f) $3 \ \square \ 6$

g) $-1 \ \square \ -7$

h) $4 \ \square \ -2$

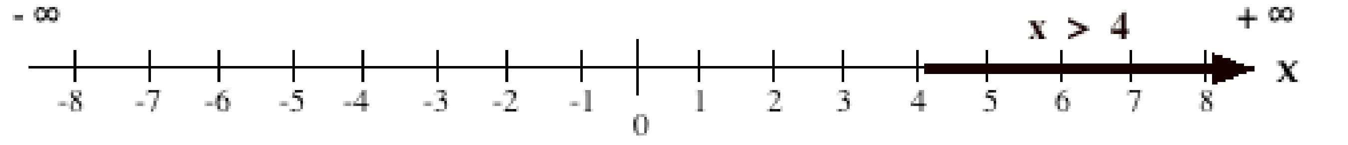
i) $0 \ \square \ -5$

## GRAPHING ON THE NUMBER LINE

Another way that we can visualize inequalities is to graph them on the number line. This will happen when we use one of the four inequality symbols, $>$, $<$, $\leq$ or $\geq$, to compare a variable to a number. It is important to note that when we consider, for example, $x > 4$, we need to consider *all* real numbers greater than 4, not just the integers.

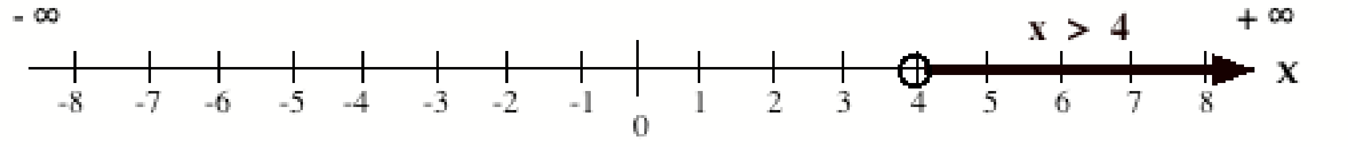
For example, if the price of a hat is more than \$4.00, such as $\mathbf{h > 4.00}$, we're not restricted to only dollar amounts, like \$5.00 or \$6.00, etc. We need to consider prices that contain pennies as well. Prices such as \$4.01, \$4.26, \$5.33, etc. In other words, we need to consider all of the numbers *between* the integers as well.

For this reason, when we graph $x > 4$, we need to show that *every* number greater than 4 is included. Below is that graph with <u>one slight error</u>. It shows, using a thicker line, all of those values along the number line that are greater than 4. Notice also the variable x is in place to the right of the graph. ∞



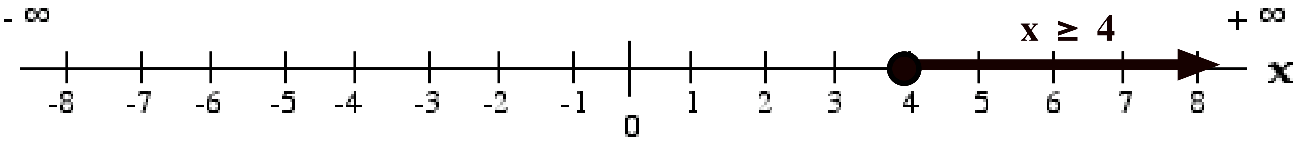
The one slight error is that the graph comes so close to touching 4 that it's hard to tell whether the value of 4 should be included or not. Certainly, every other number that the **thick line** touches is greater than 4, and that includes numbers that we can see: 5, 6, 7 and 8; numbers that we can't see: 9, 10, 11, and so on; and numbers that we can see but don't have room to write out: 4.01, 4.57, 5.08, 5.99, 6.86 and so on.

What about 4? Should it be included? That's like asking, "Is 4 greater than itself?" The answer is "No. 4 should not be included in the thick line." So, how should the graph be drawn so that we are certain that 4 is not included? Look below.

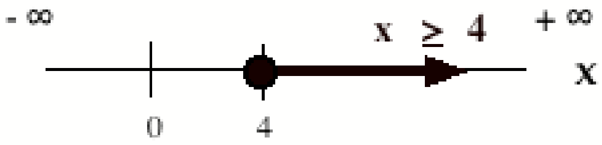


Notice the "open" circle at 4. We use this to indicate that 4 is *not included.* All of the values just barely to the right, and beyond, are included, but not 4.

How would it look if we wanted to include 4 in the graph? First, the inequality would have to allow for the variable to be equal to 4; in other words, we'd need to have a different inequality, namely $x \geq 4$. Second, we want to include 4 so we use a "closed" circle.



Actually, the graph, above, has a lot more than you really need. Though it's good to see all of the integers there, we really need to see two: 0, the *origin*, as a point of reference, and the "starting point" 4. Still show the variable and the infinities. A simplified graph looks like this:



The advantage of including so few numbers is that we're now not restricted to inequalities between just - 8 and 8. You'll see this advantage in the next example.

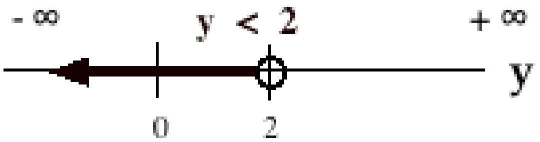
Of course, the other two inequality symbols, $<$ and $\le$, graph in a similar fashion.

**Example 3:** Graph each of these inequalities on the number line.

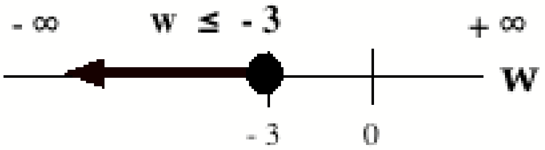
a) $y < 2$ b) $w \le -3$ c) $z > -15$

**Answer:** Be sure to include the variable and the infinities.

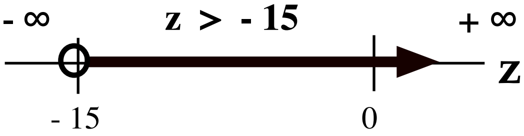
a) $y < 2$



b) $w \le -3$



c) $z > -15$



**Exercise 3:** Graph each of these inequalities on the number line provided below each one. Be sure to include the variable and the infinities, along with the origin and the graph.

a) $x > -4$

b) $y \le 6$

c) $p \ge -7$

d) $w < 2$

e) $x \leq -3$

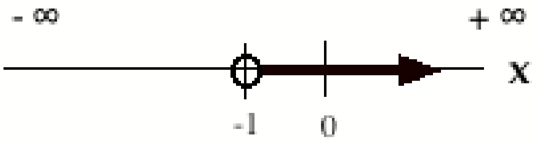
f) $y > 9$

g) $p < -10$

h) $w \geq 20$

**Example 4:** Given each graph, write an algebraic statement using one of the inequality symbols.

a)



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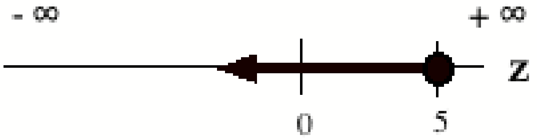
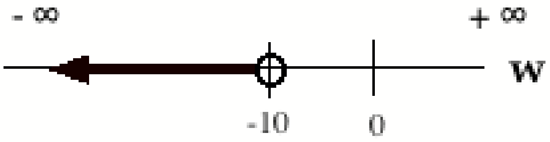
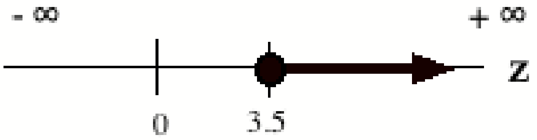


c)



d)



**Answer:** First, be sure to notice the variable at the right of the number line. These must be used correctly. Without re-drawing the graphs, the inequality statements are:

a) $x > -1$

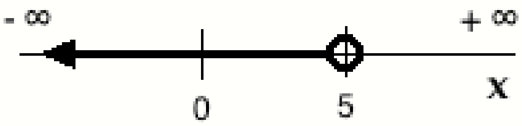
b) $z \leq 5$

c) $w < -10$

d) $z \geq 3.5$

**Exercise 4:** Given each graph, write an algebraic statement using one of the inequality symbols.

a)



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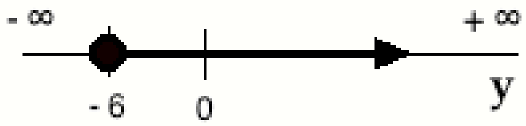

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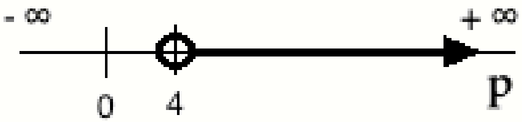

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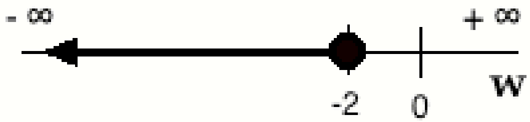

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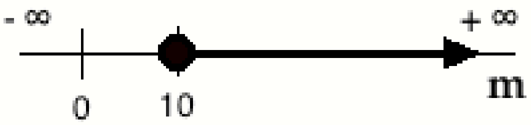

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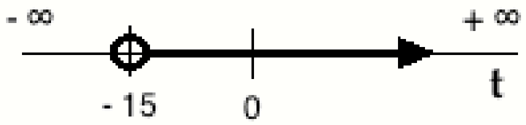

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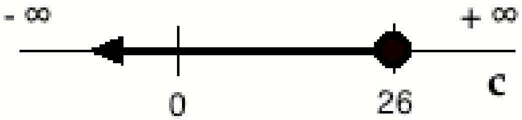

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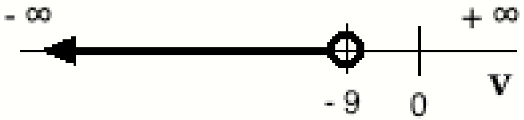

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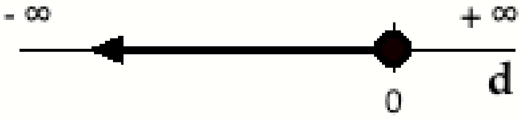

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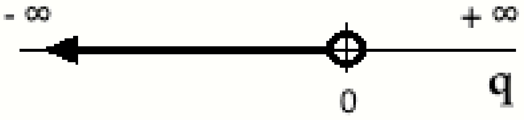

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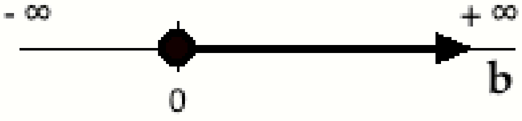

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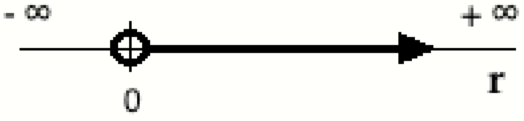


## SWITCHING SIDES OF AN INEQUALITY

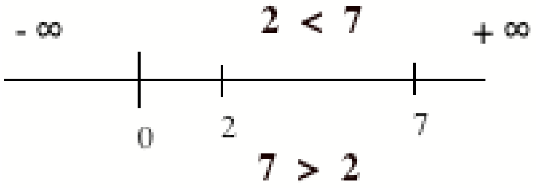
As has been mentioned, the direction of the inequality sign can helpful in determining direction on the number line. “Less than” *points* to the left and “greater than” *points* to the right. This can be confusing, though, if the variable is on the right side of the inequality.

First notice that both of these statements are true:

(1) 2 is <u>less than</u> 7: $2 < 7$

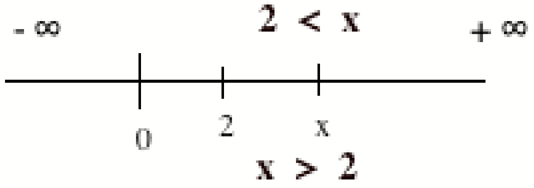
(2) 7 is <u>greater than</u> 2: $7 > 2$

Note that when the numbers “switch sides,” the inequality sign changed *direction* from "less than” to “greater than.”



It should be easy to see that both statements are true, easy because they are numbers and it couldn’t be any other way. However, if one of them were a variable, we couldn’t say with certainty that, for example, $2 < x$ is true.

We could interpret $2 < x$ as “2 is less than x.” At the same time, though, if 2 is <u>less than</u> x—if 2 is *to the left of* x—then we can just as easily say that x is to the right of 2—that “x is greater than 2.”



In other words, if we let the x and the 2 “switch sides,” as was done with the 2 and the 7, then we must also change the direction of the inequality sign:

(1) 2 is <u>less than</u> x: $2 < x$

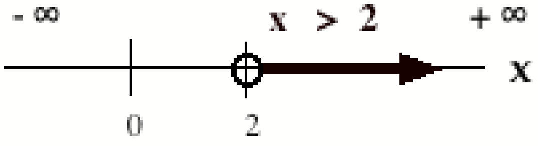
(2) x is <u>greater than</u> 2: $x > 2$

In “switching sides,” the inequality sign must change *direction* from “less than” to “greater than.”

Of course, **x** is not a specific value; it represents all of those numbers that are greater than 2.

Why is it important to recognize this? Since it’s been mentioned that the direction of the inequality sign indicates the direction of the graph—along the number line—it would be inappropriate to see $2 < x$ and graph “to the left” of 2. As you look at this closely, it says that **2 must be less than the values we graph**.

That means that 2 must be less than numbers like 2.09, 2.85, 3, 4, 5.7, etc. In other words, you must graph *to the right* of 2.



Therefore, as stated above, and as we see on the graph, $2 < x$ is the same as $x > 2$; however, the most clear way to write this is as $\mathbf{x > 2}$ because the *direction* of the inequality is more clear.

**Example 5:** (<u>with solutions shown directly below</u>) Write an equivalent statement for the inequality by “switching sides” *and* changing the “direction” of the inequality sign.

| | | | | | |
|---|---|---|---|---|---|
| a) | $3 > y$ | b) | $5 \le w$ | c) | $-3 \ge k$ |
| becomes | $y < 3$ | becomes | $w \ge 5$ | becomes | $k \le -3$ |

**Exercise 5:** Below each given inequality, write an equivalent statement by “switching sides” *and* changing the “direction” of the inequality sign.

| | | | | | |
|---|---|---|---|---|---|
| a) | $2 < x$ | b) | $4 > w$ | c) | $7 \ge m$ |
| d) | $6 \le x$ | e) | $-3 \ge p$ | f) | $-8 < c$ |
| g) | $-10 > q$ | h) | $-1 \le v$ | i) | $0 < x$ |
| j) | $0 \ge y$ | k) | $0 > n$ | l) | $0 \le b$ |

## SOLVING INEQUALITY STATEMENTS: THE SOLUTIONS

Solving means, basically, finding the solution. In a linear equation, the solution is the number that makes the equation true. For example, in $2x + 7 = 1$, the solution is $x = -3$. This is the only number that will make the statement (the equation) true. This is demonstrated by putting the value $-3$ in for x in the equation:

$$x = -3: \qquad \begin{aligned} 2x + 7 &= 1 \\ 2(-3) + 7 &= 1 \\ -6 + 7 &= 1 \\ 1 &= 1 \quad \text{True! So the solution set is } x = -3. \end{aligned}$$

In an inequality, a **solution** is still a “number that makes the statement (the inequality) true.” However, there are many more solutions than just one. In fact, there are an infinite number of solutions, too numerous to mention, so we use an inequality statement, such as $x > 2$.

To determine whether a number is a solution for $5x - 7 > 3x - 3$, for example, you might choose a number of values of x and try them all. You might find that $x = 8$ is a solution, but $x = -2$ is not:

**x = 8:**

$$\begin{aligned} 5x - 7 &> 3x - 3 \\ 5(8) - 7 &> 3(8) - 3 \\ 40 - 7 &> 24 - 3 \\ 33 &> 21 \end{aligned}$$

$$\boxed{\text{True}}$$

**x = - 2:**

$$\begin{aligned} 5x - 7 &> 3x - 3 \\ 5(-2) - 7 &> 3(-2) - 3 \\ -10 - 7 &> -6 - 3 \\ -17 &> -9 \end{aligned}$$

$$\boxed{\text{False}}$$

**Example 6:** Decide whether the given values of x make the inequality statement true or false.

$$3 - 2x \le 3x + 23$$

a) $x = 2$  b) $x = -6$  c) $x = -4$

**Answer:** Substitute the value for each x in the inequality statement.

a) **x = 2:**

$$\begin{aligned} 3 - 2(2) &\le 3(2) + 23 \\ 3 - 4 &\le 6 + 23 \\ -1 &\le 29 && \text{True!} \end{aligned}$$

b) **x = - 6:**

$$\begin{aligned} 3 - 2(-6) &\le 3(-6) + 23 \\ 3 + 12 &\le -18 + 23 \\ 15 &\le 5 && \text{False!} \end{aligned}$$

c) **x = - 4:**

$$\begin{aligned} 3 - 2(-4) &\le 3(-4) + 23 \\ 3 + 8 &\le -12 + 23 \\ 11 &\le 11 && \text{True!} \end{aligned}$$

(c) is true even though 11 is not less than itself, this inequality allows for the number, as one option, to be *equal* to 11, and it certainly is.

**Example 7:** Decide whether $w = 7$ makes $3w + 5 > 4w - 2$ true or false.

**Answer:** **w = 7:**

$$\begin{aligned} 3(7) + 5 &> 4(7) - 2 \\ 21 + 5 &> 28 - 2 \\ 26 &> 26 && \text{False!} \end{aligned}$$

Here, 26 is not *greater* than itself, and there is <u>not</u> an option for it to be equal, therefore, $w = 7$ makes the inequality statement false.

**Exercise 6:** Decide whether the given values of x make the inequality statement true or false. **SHOW ALL WORK!** Use Examples 6 and 7 as a guide.

**Inequality:** $3x - 6 < 9 - 2x$

a) $x = 6$

b) $x = -2$

c) $x = 0$

d) $x = 4$

e) $x = -1$

f) $x = 3$

You could find many more numbers that are, or are not, solutions, but using this method, how would you determine where the set of solutions “begins”? In other words, there has to be a starting value so that you can define the solution as an inequality, such as $x < 3$ (the real solution).

To answer that question, please turn to the Section 2.6, Solving Linear Inequalities.

# Answers to each Exercise

## Section 2.5

**Exercise 1**

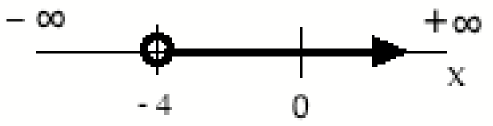
a) $x < 15$  b) $x \leq 23$  c) $x \geq 20$  d) $x \leq 8$
e) $x > 5$  f) $x < 8$

**Exercise 2**

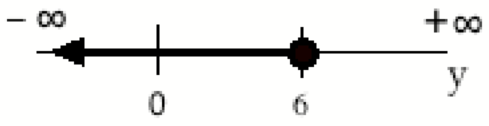
a) $8 > 3$  b) $7 > -8$  c) $-6 < 2$  d) $-4 < 9$
e) $-14 < -3$  f) $3 < 6$  g) $-1 > -7$  h) $4 > -2$
i) $0 > -5$

**Exercise 3**

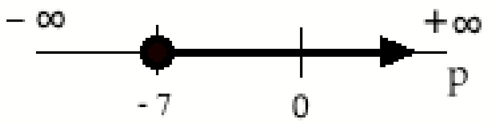
a) $x > -4$



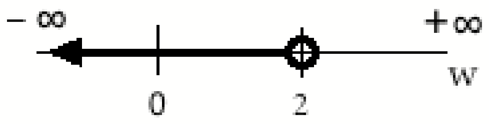
b) $y \leq 6$



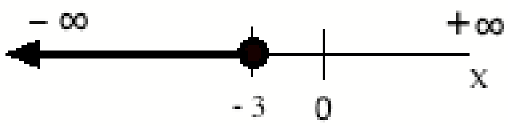
c) $p \geq -7$



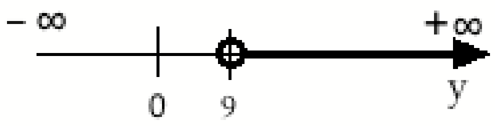
d) $w < 2$



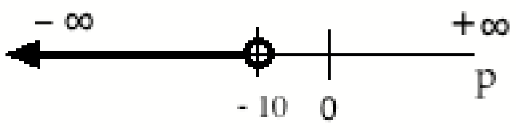
e) $x \leq -3$



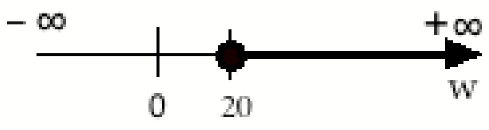
f) $y > 9$



g) $p < -10$



h) $w \geq 20$



**Exercise 4**

a) $x < 5$  b) $y \geq -6$  c) $p > 4$  d) $w \leq -2$

e) $m \geq 10$  f) $t > -15$  g) $c \leq 26$  h) $v < -9$

i) $d \leq 0$  j) $q < 0$  k) $b \geq 0$  l) $r > 0$

**Exercise 5**

a) $x > 2$  b) $w < 4$  c) $m \leq 7$  d) $x \geq 6$

e) $p \leq -3$  f) $c > -8$  g) $q < -10$  h) $v \geq -1$

i) $x > 0$  j) $y \leq 0$  k) $n < 0$  l) $b \geq 0$

**Exercise 6**

a) $12 < -3$; False  b) $-12 < 13$; True  c) $-6 < 9$; True

d) $6 < 1$; False  e) $-9 < 11$; True  f) $3 < 3$; False

## Section 2.5 Focus Exercises

**1.** Translate each of these sentences into algebra. For each, Let x = the number of cars.

a) Janet has owned more than 12 cars in her lifetime. ______________

b) There were at least 30 cars in the parking lot. ______________

c) There were no more than 6 cars parked in front of Joe's house. ______________

d) There were fewer than 10 cars following the wedding couple's limo. ______________

**2.** Fill in the box with an inequality sign (either < or >) that makes the statement true. (You may use the number line, above, to help you think about the answers.)

a) - 9 ☐ 4
b) - 8 ☐ - 9
c) - 7 ☐ 3

d) - 5 ☐ 99
e) - 25 ☐ - 4
f) 4 ☐ - 7

g) 2 ☐ - 8
h) - 5 ☐ - 3
i) 0 ☐ - 6

**3.** Graph each of these inequalities on the number line provided below each one. Be sure to include the variable and the infinities, along with the origin and the graph.

**a)** $x \le 4$

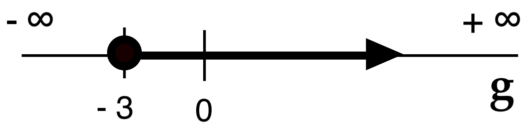
**b)** $y < -1$

**c)** $p > -3$

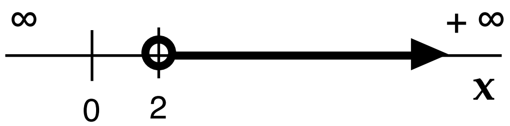
**d)** $w \ge 0$

**4.** Given each graph, write an algebraic statement using one of the inequality symbols.

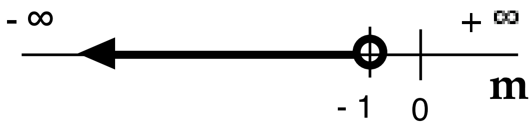
a)



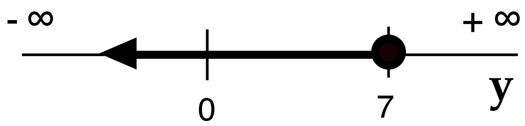
b)



c)



d)



**5.** Below each given inequality, write an equivalent statement by "switching sides" *and* changing the "direction" of the inequality sign.

a) $-6 \geq y$

b) $-1 \leq r$

c) $0 < w$

d) $0 > h$

e) $4 < n$

f) $5 \geq x$

**6.** Decide whether the given values of x make the inequality statement true or false. SHOW ALL WORK!

**Inequality:** $4 - x \geq 3x - 8$

a) $x = 6$

b) $x = -2$

c) $x = 0$

d) $x = 4$

e) $x = -1$

f) $x = 3$