Section 5.3 Factor By Grouping

INTRODUCTION

In the previous section you were introduced to factoring out a common monomial factor from a polynomial. For example, in the binomial

$$6x^2 + 15x,$$

we can recognize a common factor of **3x** in each of the terms. We could even write the binomial as

$$\mathbf{3x}\cdot 2x + \mathbf{3x}\cdot 5,$$

and we know we can factor out the *common monomial factor* of **3x**

to the left:	or to the right:
$\mathbf{3x}(2x + 5)$	$(2x + 5)\mathbf{3x}$.

FACTORING COMMON BINOMIALS

We can factor in a similar manner if the common factor is not a monomial. Consider

$$\mathbf{(A)}\cdot 2x + \mathbf{(A)}\cdot 5$$

(compare this with $\mathbf{3x}\cdot 2x + \mathbf{3x}\cdot 5$, above.)

where **(A)** represents **A**ny polynomial—maybe a monomial, a binomial or a trinomial. Since the same factor **(A)** is common to both terms, it can be factored out either

to the left:	or to the right:
$\mathbf{(A)}(2x + 5)$	$(2x + 5)\mathbf{(A)}$.

Example 1: Factor out the common binomial factor. State the factors when complete.

a) $\mathbf{(4y - 3)}\cdot 2y^3 + \mathbf{(4y - 3)}\cdot 7$ b) $5x^2\cdot\mathbf{(x^3 + 5)} - 9\cdot\mathbf{(x^3 + 5)}$

Procedure: Recognize the common binomial factor and factor it out *either to the left or to the right*.

a)	b)
$\mathbf{(4y - 3)}\cdot(2y^3 + 7)$	$(5x^2 - 9)\mathbf{(x^3 + 5)}$
or $(2y^3 + 7)\cdot\mathbf{(4y - 3)}$	or $\mathbf{(x^3 + 5)}(5x^2 - 9)$
<u>The factors are</u> $(4y - 3)$ and $(2y^3 + 7)$.	<u>The factors are</u> $(5x^2 - 9)$ and $(x^3 + 5)$.

Exercise 1 Factor out the common binomial factor. State the factors when complete.

a) $(3x + 5)\cdot x^2 + (3x + 5)\cdot 8$

b) $7c\cdot(c^2 - 2) - 4\cdot(c^2 - 2)$

c) $w(5x - 2y) + 1\cdot(5x - 2y)$

d) $a^2\cdot(a + 2b) - 3b\cdot(a + 2b)$

e) $(3 - 2m)\cdot 9 - (3 - 2m)\cdot m$

f) $3x\cdot(x + 2) - 1\cdot(x + 2)$

FACTOR BY GROUPING

Sometimes, a **four-term polynomial** can be factored using a method called **factor by grouping**.

Factor by grouping requires that we use parentheses to break the four-term polynomial into two groups of two terms each; in other words, each grouping will be a binomial. We *usually* group the first two terms separately from the last two terms.

For example, $5x^3 + 15x^2 + 2x + 6$

can be grouped as $(5x^3 + 15x^2) + (2x + 6)$.

(Notice that the plus sign (+) remains between the groupings.)

Since we are interested in factoring the polynomial into a product of some sort, simply putting a plus sign between two groupings is not enough. We'll actually need to factor out a common factor, as in Example 1. Therefore, our goal is to see if we can work with the two groupings, individually, to find a common factor within.

The strategy is to factor the groupings as best possible. That means that we'll need to look for a common *monomial* factor within each grouping. In fact, *factor by grouping* requires that at least one binomial grouping has an obvious common *monomial* factor.

For example, in

$(5x^3 + 15x^2) + (2x + 6)$ — the first pair has a common factor of $\mathbf{5x^2}$;
the second pair has a common factor of **2**.

Factoring out those common factors, we get:

$$(5x^3 + 15x^2) + (2x + 6)$$

$$= \mathbf{5x^2}(x + 3) + \mathbf{2}(x + 3)$$

Notice that, as each grouped binomial has been factored,

(1) there's still a plus sign between the groupings, and

(2) each has a common *binomial* factor $(x + 3)$.

At this point, the whole four-term polynomial has *not* been factored; we've only factored the individual groupings. Recognizing the factors of the individual groupings we see another common factor.

The factors of the first grouping are	The factors of the second grouping are
$5x^2$ and $\mathbf{(x + 3)}$	2 and $\mathbf{(x + 3)}$

You should notice that $\mathbf{(x + 3)}$ is a common factor of both groupings; in other words, it is a common *binomial* factor. This means that we can now factor the entire polynomial; we can factor out the quantity $\mathbf{(x + 3)}$, just as we did in Example 1.

we can factor $5x^2\mathbf{(x + 3)} + 2\mathbf{(x + 3)}$ as either

(1) $\mathbf{(x + 3)}(5x^2 + 2)$ or (2) $(5x^2 + 2)\mathbf{(x + 3)}$

Example 2: Use *factor by grouping* to factor each four-term polynomial.

a) $3ax + 4a + 3bx + 4b$

b) $2x^3 - 10x^2 + 3x - 15$

Procedure:

	a)	b)
	$3ax + 4a + 3bx + 4b$	$2x^3 - 10x^2 + 3x - 15$
Create two groups, separated by a +	$= (3ax + 4a) + (3bx + 4b)$	$= (2x^3 - 10x^2) + (3x - 15)$
factor each grouping:	$= a\mathbf{(3x + 4)} + b\mathbf{(3x + 4)}$	$= 2x^2\cdot\mathbf{(x - 5)} + 3\mathbf{(x - 5)}$
factor out the binomial factor	$= (a + b)\mathbf{(3x + 4)}$	$= (2x^2 + 3)\mathbf{(x - 5)}$
	or $\mathbf{(3x + 4)}(a + b)$	or $\mathbf{(x - 5)}(2x^2 + 3)$
	The factors are $(a + b)$ and $(3x + 4)$.	The factors are $(2x^2 + 3)$ and $(x - 5)$.

If one of the groupings doesn't *appear* to have a common factor, then the common factor is most likely just 1 (one) since 1 is a factor of every term.

Example 3: Use *factor by grouping* to factor this four-term polynomial.

$$6y^3 - 9y^2 + 2y - 3$$

Procedure: The polynomial can be grouped as $(6y^3 - 9y^2) + (2y - 3)$.

When looking for the obvious common factors, the first pair gives $\mathbf{3y^2}$, but the second pair gives just **1**:

$$(6y^3 - 9y^2) + (2y - 3)$$

$$= \mathbf{3y^2}(2y - 3) + \mathbf{1}(2y - 3)$$

The common binomial factor is (2y – 3):

$$= 3y^2\mathbf{(2y - 3)} + 1\mathbf{(2y - 3)}$$

Rewriting this step is unnecessary but shown for emphasis.

$$= (3y^2 + 1)\mathbf{(2y - 3)}$$

The factors are $(3y^2 + 1)$ and $(2y - 3)$

Exercise 2 Use *factor by grouping* to factor each four-term polynomial.

a) $2x^3 + 14x^2 + 3x + 21$

b) $6ab^2 - 6b^2 + 5a - 5$

c) $10x^3 + 8x^2 + 5x + 4$

d) $10y^3 - 15y^2 + 2y - 3$

Switching the Middle Terms

If a four-term polynomial is factorable using "factor by grouping," then we can usually switch the order of the two middle terms (the commutative property of addition allows us to do that), and the polynomial can still be factored using "factor by grouping."

Let's apply this to the first example:

Factor $5x^3 + 15x^2 + 2x + 6$ by first switching the middle two terms:

	$5x^3 + \mathbf{15x^2} + 2x + 6$	start here
=	$5x^3 + 2x + \mathbf{15x^2} + 6$	switch the second and third terms
=	$(5x^3 + 2x) + (15x^2 + 6)$	create the two groupings
=	$x(5x^2 + 2) + 3(5x^2 + 2)$	take out the common factor from each grouping
=	$(x + 3)(5x^2 + 2)$ or $(5x^2 + 2)(x + 3)$	Either answer is okay.

Again, the factors are $(x + 3)$ and $(5x^2 + 2)$

If we try switching the middle two terms of $6y^3 - 9y^2 + 2y - 3$

we get $6y^3 + 2y - 9y^2 - 3$

At this point, *we must be careful*. Notice that the third term is now being subtracted, it is *negative*. When the third term is negative, we must treat it as "adding the opposite." In other words, we should treat the third term as **+** $-9y^2$. This is so that we may maintain a **+** between the two groupings:

$$6y^3 + 2y \mathbf{+} -9y^2 - 3.$$

Now when we create the groups, there will still be a plus sign between them:

$$(6y^3 + 2y) \mathbf{+} (-9y^2 - 3).$$

In this case the common *monomial* factors are **2y**, from the first group, and **-3**, from the second group:

	$\mathbf{2y}(3y^2 + 1) + \mathbf{-3}(3y^2 + 1)$	We factor out the negative along with the 3 because the first term, $-9y^2$, is negative.
=	$(2y \mathbf{+} \mathbf{-}3)(3y^2 + 1)$	Whenever the first term of a binomial is negative, we factor out the negative along
or just	$(2y \mathbf{-} 3)(3y^2 + 1)$	with any other common factor.

From the start, the factor by grouping process looks like this:

$$6y^3 - 9y^2 + 2y - 3$$
$$= 6y^3 + 2y - 9y^2 - 3 \quad \text{switch the second and third terms}$$
$$= 6y^3 + 2y + -9y^2 - 3 \quad \text{change the third term to "add the opposite"}$$
$$= (6y^3 + 2y) + (-9y^2 - 3) \quad \text{create the two groupings}$$
$$= 2y(3y^2 + 1) + -3(3y^2 + 1) \quad \text{take out the common factor from each grouping}$$
$$= (2y - 3)(3y^2 + 1) \quad \text{or} \quad (3y^2 + 1)(2y - 3)$$

The factors are $(2y - 3)$ and $(3y^2 + 1)$.

This four-term polynomial was easiest to factor when left the way it was originally presented—with a *positive* third term. This illustrates a point: if possible, it's better to have the third term be positive, so if it isn't already, see if switching the two middle terms will help.

Example 4: Switch the middle two terms, then use factor by grouping.

a) $3x^3 + 12x^2 - 2x - 8$

$= 3x^3 - 2x + 12x^2 - 8$

$= (3x^3 - 2x) + (12x^2 - 8)$

$= x(3x^2 - 2) + 4(3x^2 - 2)$

$= (x + 4)(3x^2 - 2)$

or $(3x^2 - 2)(x + 4)$

The factors are $(x + 4)$ and $(3x^2 - 2)$.

b) $4x^3 - 8x^2 - 5x + 10$

$= 4x^3 - 5x - 8x^2 + 10$

$= (4x^3 - 5x) + (-8x^2 + 10)$

$= x\cdot(4x^2 - 5) + -2(4x^2 - 5)$

$= (x - 2)(4x^2 - 5)$

or $(4x^2 - 5)(x - 2)$

The factors are $(4x^2 - 5)$ and $(x - 2)$.

Exercise 3 Switch the middle two terms, then use factor by grouping.

a) $2x^3 + 14x^2 + 3x + 21$

b) $10y^3 - 15y^2 + 2y - 3$

In this next example the third term is negative, but so is the second term, so it won't do much good to switch the two middle terms:

$$4x^3 - x^2 - 24x + 6 \qquad \text{(This is left as an exercise for the student.)}$$

Note that it is never suggested switching any other terms, just the middle two. This is rather consistent except for one other situation: if a four-term polynomial is not already in **descending order**, then write it so. For example,

if given $4x^2 - 5x - 20 + x^3$ to factor, first rewrite it in descending order as
$x^3 + 4x^2 - 5x - 20$ (This, too, is left as an exercise for the student.)

Example 5: Use *factor by grouping* to factor these polynomials.

a) $12x - 2x^2 + 3x^3 - 8$

First, descending order: $3x^3 - 2x^2 + 12x - 8$

$= (3x^3 - 2x^2) + (12x - 8)$

$= x^2(3x - 2) + 4(3x - 2)$

$= (x^2 + 4)(3x - 2)$

or $(3x - 2)(x^2 + 4)$

The factors are $(x^2 + 4)$ and $(3x - 2)$.

b) $10 - 5x + 4x^3 - 8x^2$

$4x^3 - 8x^2 - 5x + 10$

$= (4x^3 - 8x^2) + (-5x + 10)$

$= 4x^2 \cdot (x - 2) + -5(x - 2)$

$= (4x^2 - 5)(x - 2)$

or $(x - 2)(4x^2 - 5)$

The factors are $(4x^2 - 5)$ and $(x - 2)$.

Exercise 4 Use *factor by grouping* to factor each four-term polynomial.

a) $-2m - 12m^2 + 3m^3 + 8$

b) $10y^2 - 5 + 4y^3 - 2y$

Guidelines for Factor by Grouping

The goal in this process is to get two groupings; each grouping will have two terms (a binomial). If the polynomial is factorable, then at least one of the binomials will have an obvious common monomial factor. The following guidelines are given so that you can be consistent in your work.

1. If necessary—and if possible—write the four-term polynomial in descending order, even if the lead term is negative.

Example 1:

If you're given	$5x^3 - 4 + 2x - 10x^2$
then rewrite it as	$5x^3 - 10x^2 + 2x - 4$

2. If necessary, rearrange the two middle terms so that the third term is positive. If this isn't possible (because both middle terms are negative), then write the third term as "plus the opposite" of the third term.

	Example 2:	**Example 3:**
If you're given:	$5x^3 + 10x^2 - 2x - 4$	$5x^3 - 10x^2 - 2x + 4$
then rewrite it as:	$5x^3 - 2x + 10x^2 - 4$	$5x^3 - 10x^2 + -2x + 4$

3. Group the first two terms and group the last two terms and separate the groupings by a plus sign.

	Example 2:	**Example 3:**
If you're given:	$5x^3 - 2x + 10x^2 - 4$	$5x^3 - 10x^2 + -2x + 4$
then rewrite it as:	$(5x^3 - 2x) + (10x^2 - 4)$	$(5x^3 - 10x^2) + (-2x + 4)$

4. Factor each grouping, individually, by finding the greatest common monomial factor; if the lead term in either grouping is negative, then factor out the negative along with the common factor. If there is no apparent common factor, then use 1 (or -1 if the lead term is negative).

5. When step #4 is complete, the two resulting quantities (binomials) should be identical in every way. **If they are not,** then either

   i) an algebraic mistake has been made somewhere along the way; or

   ii) the terms of the polynomial should be paired and grouped differently.

   In any case, find the mistake or the proper pairings and correct *all* of your work. When the binomials are exactly the same, then they can be treated as the "new" common factor and be "factored out." They can be factored out "to the left" or "to the right." The result will be a product of two binomials.

6. Check your answer by multiplying the binomials using the FOIL method.

7. Check to see if either binomial can be factored further.

Exercise 5 Use *factor by grouping* to factor each four-term polynomial. Use the guidelines set forth on the previous page.

a) $4x^3 + 3x^2 + 20x + 15$

b) $15c^2 + 6c - 5cd - 2d$

c) $20 - 24m^2 + 6m^3 - 5m$

d) $6xy - 8x + 3y - 4$

Exercise 6 The following have middle terms that are like terms, but DO NOT COMBINE THE LIKE TERMS in these exercises. Instead, use *factor by grouping* to factor each one.

a) $3x^2 + 12x - 2x - 8$

b) $10x^2 - 8x - 5x + 4$

In a little bit you'll be asked to factor trinomials using the method of "Factor By Grouping." Before you learn that, though, you need to be reminded of the Factor Game, first presented in Section 1.5. Please take a little time to review Section 1.5 before moving on in this section.

THE FACTOR GAME

RULES:

1. You are given two numbers: the **Key number** (Key #) and the **Sum number** (Sum #);

2. you are to find a **factor pair** of the Key # that will **add** up to the Sum #. This factor pair is called the **solution**.

Special Notes: i) It's actually possible that there is no solution.

ii) If there is a solution, there will be only *one* solution (two numbers—a factor pair) for each set of Key and Sum numbers.

Example 6: Find the solution for each Key # and Sum # as shown.

a) Key # = 15 and Sum # = 8

You can probably see right away, without any work that the solution is: **3 and 5**.

Let's check: $3 \cdot 5 = 15$ (Key #) and $3 + 5 = 8$ (Sum #).

b) Key # = 60 and Sum # = - 19. Because the Key # is positive and the Sum # is negative, it must be that each factor of the factor pair is negative.

This is going to take a little more work as the solution probably isn't as obvious. Let's generate a *partial* list of factor pairs for 60; keep going until you find the right pair:

Factor pairs of: + 60

+ 60		Sum	
/	\		
- 1	- 60	- 61	too large
- 2	- 30	- 32	still too large, but closer
- 3	- 20	- 23	closer still
- 4	- 15	- 19	THIS IS IT!

There are more factor pairs of 60, but we don't need to look at them because we've already found the solution: **- 4 and - 15**.

c) Key # = - 12 and Sum # = 1. One factor is positive and the other is negative.

Factor pairs of:	- 12		**<u>Sum</u>**	(Think of the sum number as being +1.)
	/	\		
	- 4	+ 3	- 1	The sum needs to be positive, not negative.
Try again:	+ 4	- 3	+ 1	This is it.

The solution is: **<u>+4 and - 3</u>**.

d) Key # = 30 and Sum # = 12

This is one of those (rare) times when there simply is **<u>no solution</u>**. There is no factor pair of 30 for which the sum is 12.

<u>Exercise 7</u> Find the solution for the given Key number and Sum number.

a) Key # = 24, Sum # = 10:

b) Key # = 18, Sum # = 11:

c) Key # = 60, Sum # = - 23:

d) Key # = 36, Sum # = 12:

e) Key # = 30, Sum # = - 11:

f) Key # = 16, Sum # = - 8:

g) Key # = - 36, Sum # = 5:

h) Key # = - 45, Sum # = - 4:

i) Key # = - 16, Sum # = 0:

j) Key # = - 30, Sum # = - 13:

k) Key # = - 60, Sum # = - 11:

l) Key # = - 36, Sum # = - 9:

Answers to each Exercise

Section 5.3

Exercise 1:	a)	$(3x + 5)(x^2 + 8)$	b)	$(7c - 4)(c^2 - 2)$
	c)	$(w + 1)(5x - 2y)$	d)	$(a^2 - 3b)(a + 2b)$
	e)	$(3 - 2m)(9 - m)$	f)	$(3x - 1)(x + 2)$
Exercise 2:	a)	$(2x^2 + 3)(x + 7)$	b)	$(6b^2 + 5)(a - 1)$
	c)	$(2x^2 + 1)(5x + 4)$	d)	$(5y^2 + 1)(2y - 3)$
Exercise 3:	a)	$(x + 7)(2x^2 + 3)$	b)	$(2y - 3)(5y^2 + 1)$
Exercise 4:	a)	$(3m^2 - 2)(m - 4)$	b)	$(2y^2 - 1)(2y + 5)$
Exercise 5:	a)	$(x^2 + 5)(4x + 3)$	b)	$(3c - d)(5c + 2)$
	c)	$(6m^2 - 5)(m - 4)$	d)	$(2x + 1)(3y - 4)$
Exercise 6:	a)	$(3x - 2)(x + 4)$	b)	$(2x - 1)(5x - 4)$

Exercise 7:	a)	+ 6 and + 4	b)	+ 9 and + 2	c)	- 20 and - 3
	d)	+ 6 and + 6	e)	- 6 and - 5	f)	- 4 and - 4
	g)	+ 9 and - 4	h)	- 9 and + 5	i)	+ 4 and - 4
	j)	- 15 and + 2	k)	- 15 and + 4	l)	- 12 and + 3

Section 5.3 Focus Exercises

1. Use *factor by grouping* to factor each four-term polynomial. (Use the techniques learned in this section to make factoring the polynomial easiest for you.) State the factors when complete.

a) $2x^3 + 7x^2 + 8x + 28$

b) $m^3 + 6m^2 - 2m - 12$

c) $3x^2 - 4xy - 15x + 20y$

d) $6x^3 + 3x^2 - 2x - 1$

e) $6r - 5pr + 6p - 5p^2$

f) $15y - 8y^2 - 12 + 10y^3$

g) $6x - 8 - 3x^3 + 4x^2$

h) $20x^2 - 3x - 12 + 5x^3$

2. Find the solution for the given Key number and Sum number. If there is no solution, state so.

a) Key # = 20, Sum # = 12:

b) Key # = 25, Sum # = - 10:

c) Key # = 12, Sum # = 13:

d) Key # = 40, Sum # = - 13:

e) Key # = 42, Sum # = 1:

f) Key # = 28, Sum # = - 16:

g) Key # = - 60, Sum # = 4:

h) Key # = - 42, Sum # = - 1:

i) Key # = - 9, Sum # = 0:

j) Key # = - 20, Sum # = 9:

k) Key # = - 24, Sum # = - 5:

l) Key # = - 15, Sum # = - 14:

m) Key # = - 45, Sum # = 12:

n) Key # = 30, Sum # = - 17: