Section 2.5 Ratios and Proportions

Objectives
In this section, you will learn to:

- Apply cross-multiplication to to determine equivalent ratios.
- Define and identify *proportions*.
- Solve proportions.
- Use proportions in problem solving.

To successfully complete this section, you need to understand:

- Simplifying fractions (R.3)
- Solving equations (2.1, 2.2, & 2.3)
- Writing a legend (2.4)

INTRODUCTION: RATIOS

A **ratio** between two numbers *compares their relative size*, using division. This relative size will not change, no matter how large or small the numbers may get.

> The **ratio** of two numbers, A and B, is written as either A **:** B or $\frac{A}{B}$.
> In words, this is read, "**A to B**."

Consider this: Two brothers, Kenny and Jimi decided to go into business together selling lemonade. Kenny, the older brother, contributed \$10 to get the business started and Jimi contributed \$6. When the profits were counted at the end of the summer, Kenny got \$75 and Jimi got \$45. Is this a fair sharing of the profits?

To answer this question, use the ratio **Kenny : Jimi** to look at the relative size of both (1) the investments each made and (2) the profits each received:

(1) Investments: \$10 **:** \$6 or $\frac{10}{6}$

(2) Profits: \$75 **:** \$45 or $\frac{75}{45}$

If these two ratios are equivalent, then the share of the profits is fair.

One way we can determine whether the ratios are equivalent is to simplify each and see if they simplify to the same ratio or fraction:

$\frac{10}{6}$ simplifies by a factor of 2 to $\frac{5}{3}$

$\frac{75}{45}$ simplifies by a factor of 15 to $\frac{5}{3}$

Because each fraction simplifies to $\frac{5}{3}$, it is safe to conclude that the sharing of profits was fair.

EQUIVALENT RATIOS: CROSS-MULTIPLICATION

When two ratios have the same value, such as $\frac{10}{6}$ and $\frac{75}{45}$, we call them **equivalent ratios**: $\frac{10}{6} = \frac{75}{45}$.

A technique for verifying whether two ratios, $\frac{a}{b}$ and $\frac{c}{d}$, are equivalent is called *cross-multiplication*. Cross-multiplication has the effect of multiplying the denominators to the opposite side numerators, as shown at right:

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$a \cdot d = b \cdot c$$

For example, we can verify that $\frac{10}{6}$ and $\frac{75}{45}$ are equivalent using cross-multiplication:

$\frac{10}{6} \overset{?}{=} \frac{75}{45}$	Multiply each denominator across the equal sign.
$10 \cdot 45 \overset{?}{=} 6 \cdot 75$	Evaluate.
$450 = 450$	True ✓

Because both products are 450, we have verified that the ratios $\frac{10}{6}$ and $\frac{75}{45}$ are equivalent.

Here is why cross-multiplication works:

$\frac{a}{b} = \frac{c}{d}$	Multiply each side by the common denominator *bd*, or $\frac{bd}{1}$.
$\frac{a}{b} \cdot \frac{\boldsymbol{bd}}{\mathbf{1}} = \frac{\boldsymbol{bd}}{\mathbf{1}} \cdot \frac{c}{d}$	On the left side, *b* divides out, and on the right side, *d* divides out.
$\frac{a}{1} \cdot \frac{\mathbf{1} \cdot \boldsymbol{d}}{\mathbf{1}} = \frac{\boldsymbol{b} \cdot \mathbf{1}}{\mathbf{1}} \cdot \frac{c}{1}$	Simplify.
$a \cdot d = b \cdot c$	Cross-multiplication!

As mentioned, cross-multiplication provides a way to verify whether two ratios are equivalent. After cross-multiplying, if the products are *not* the same, then the two ratios are not equivalent.

Example 1: Determine whether or not the ratios are equivalent by cross-multiplying.

a) $\frac{6}{4} \overset{?}{=} \frac{15}{10}$

b) $\frac{8}{20} \overset{?}{=} \frac{12}{25}$

Procedure: Multiply each denominator across the equal sign to the numerator on the opposite side.

Answer:

a) $\dfrac{6}{4} \overset{?}{=} \dfrac{15}{10}$ Cross-multiply.

$\dfrac{6}{4} \overset{?}{=} \dfrac{15}{10}$

$6 \cdot 10 \overset{?}{=} 4 \cdot 15$ Multiply.

$60 = 60$

Yes, $\dfrac{6}{4}$ and $\dfrac{15}{10}$ ***are*** equivalent.

b) $\dfrac{8}{20} \overset{?}{=} \dfrac{12}{25}$ Cross-multiply.

$\dfrac{8}{20} \overset{?}{=} \dfrac{12}{25}$

$8 \cdot 25 \overset{?}{=} 20 \cdot 12$ Multiply.

$200 \neq 240$

No, $\dfrac{8}{20}$ and $\dfrac{12}{25}$ are ***not*** equivalent.

You Try It 1 Determine whether the ratios are equivalent by cross-multiplying. Use Example 1 as a guide.

a) $\dfrac{6}{15} \overset{?}{=} \dfrac{8}{25}$

b) $\dfrac{9}{6} \overset{?}{=} \dfrac{12}{8}$

c) $\dfrac{\frac{3}{2}}{2} \overset{?}{=} \dfrac{\frac{5}{2}}{\frac{10}{3}}$

PROPORTIONS

An equation that shows the equivalence of two ratios is called a **proportion**. Based on the results in Example 1, $\dfrac{6}{4} = \dfrac{15}{10}$ is a proportion because $\dfrac{6}{4}$ and $\dfrac{15}{10}$ are equivalent ratios. However, we cannot set up a proportion between $\dfrac{8}{20}$ and $\dfrac{12}{25}$ because they are not equivalent ratios.

When a proportion contains a variable—either in one or both of the numerators or in one or both of the denominators—we can solve for that variable by cross-multiplying.

Examples of such proportions are $\dfrac{w}{8} = \dfrac{27}{12}$, $\dfrac{20}{y} = \dfrac{12}{15}$, and $\dfrac{x}{5} = \dfrac{x+16}{15}$.

Caution: If the equation involves more than just two equivalent fractions, then it is *not* a proportion and we *cannot* use cross-multiplication to clear the fractions. Here are some examples of equations with fractions and some examples of proportions.

Equations with fractions	**Proportions**
To solve each of these, we first clear the fractions by multiplying each side by the LCD.	To solve each proportion, we can use cross-multiplication.
$\frac{x}{4} + 7 = \frac{x}{3}$	$\frac{35}{10} = \frac{x}{4}$
$\frac{x+3}{8} - \frac{x}{2} = 5$	$\frac{x+2}{4} = \frac{x-1}{2}$

Example 2: Solve for x in each proportion. Verify the solution.

a) $\frac{x}{8} = \frac{3}{4}$ b) $\frac{15}{12} = \frac{10}{x}$

Procedure: Use cross-multiplication to get the equation into a more familiar form. Then solve the resulting equation. We can verify the solution by replacing x in the proportion and simplifying each fraction.

Answer: a)

$\frac{x}{8} = \frac{3}{4}$	Cross-multiply.
$x \cdot 4 = 8 \cdot 3$	Simplify each side.
$4x = 24$	Divide each side by 4.
$\frac{4x}{4} = \frac{24}{4}$	Simplify.
$x = 6$	Verify $x = 6$. ↗

<u>Verify $x = 6$:</u>

$\frac{6}{8} \stackrel{?}{=} \frac{3}{4}$

$\frac{3}{4} = \frac{3}{4}$ ✓

b)

$\frac{15}{12} = \frac{10}{x}$	Cross-multiply.
$15 \cdot x = 12 \cdot 10$	Simplify each side.
$15x = 120$	Divide each side by 15.
$\frac{15x}{15} = \frac{120}{15}$	Simplify.
$x = 8$	Verify $x = 8$. ↗

<u>You finish it:</u>

Verify that $x = 8$ is the solution.

Note: In Example 2b), we could have simplified the left side fraction first. That way, the proportion is easier to work with:

$$\frac{15}{12} = \frac{10}{x} \qquad \text{Simplify the left side by a factor of 3.}$$

$$\frac{5}{4} = \frac{10}{x} \qquad \text{Cross-multiply.}$$

$$5x = 40 \qquad \text{Divide each side by 5.}$$

$$x = 8 \qquad \text{This is the same solution as in Example 2b).}$$

You Try It 2 Solve for x in each proportion. Use Example 2 as a guide.

a) $\dfrac{2}{9} = \dfrac{x}{27}$ b) $\dfrac{10}{8} = \dfrac{15}{x}$ c) $\dfrac{x}{35} = \dfrac{24}{40}$

Sometimes proportions have variables in more than one place, such as $\dfrac{x+6}{5} = \dfrac{x}{2}$. This proportion can still be solved by cross-multiplying. First, though, we must recognize that—because the division bar is a grouping symbol—the left side numerator is a quantity:

$$\frac{(x+6)}{5} = \frac{x}{2} \qquad \text{Cross-multiply.}$$

$$\frac{(x+6)}{5} = \frac{x}{2} \qquad \text{As we cross-multiply, the parentheses remain until we distribute.}$$

$$(x+6)\cdot 2 = 5\cdot x \qquad \text{The result of cross-multiplying is a linear equation.}$$

$$2x + 12 = 5x$$

Now we solve as we would any linear equation—by isolating the variable.

This process remains true even if both variable terms are in the denominator.

Example 3: Solve for the variable in each proportion. Verify the solution.

a) $\dfrac{x}{2} = \dfrac{x+6}{5}$ b) $\dfrac{4}{w-1} = \dfrac{6}{w+2}$

Procedure: Use parentheses to group all quantities; then use cross-multiplication to create a linear equation. Solve the linear equation by isolating the variable.

Answer:

a)

$$\frac{x}{2} = \frac{x+6}{5}$$ Show parentheses in the first numerator.

$$\frac{x}{2} = \frac{(x+6)}{5}$$ Cross-multiply.

$$5 \cdot x = (x+6) \cdot 2$$ Distribute on the right side.

$$5x = 2x + 12$$ Add -2x to each side.

$$\mathbf{-2x} = \mathbf{-2x}$$

$$3x = 12$$ Divide each side by 3.

$$\frac{3x}{\mathbf{3}} = \frac{12}{\mathbf{3}}$$ Simplify.

$$x = 4$$ Verify $x = 4$. ↗

<u>You finish it:</u>

Verify that $x = 4$ is the solution.

b)

$$\frac{4}{w-1} = \frac{6}{w+2}$$ Show parentheses in each denominator.

$$\frac{4}{(w-1)} = \frac{6}{(w+2)}$$ Cross-multiply.

$$4 \cdot (w+2) = (w-1) \cdot 6$$ Distribute on each side.

$$4w + 8 = 6w - 6$$ Add -6w to each side.

$$\mathbf{-6w} = \mathbf{-6w}$$

$$-2w + 8 = -6$$ Add -8 to each side.

$$\mathbf{-8} = \mathbf{-8}$$

$$-2w = -14$$ Divide each side by -2.

$$\frac{-2w}{-2} = \frac{-14}{-2}$$ Simplify.

$$w = 7$$ Verify $w = 7$. ↗

<u>You finish it:</u>

Verify that $w = 7$ is the solution.

You Try It 3 Solve each proportion. Verify the solution. Use Example 3 as a guide.

a) $\dfrac{x+3}{5} = \dfrac{x}{2}$ b) $\dfrac{2y-2}{4} = \dfrac{3y-9}{2}$ c) $\dfrac{5}{8} = \dfrac{2w+2}{4w-4}$

To this point, we've worked only with integer values. However, many times a ratio will include fractional values in the numerator or denominator.

For example, if it takes $\frac{2}{3}$ cup of milk for 4 pancakes, then we can double the recipe to make 8 pancakes by doubling the amount of milk needed: $\frac{4}{3}$, or $1\frac{1}{3}$, cups of milk. (For our purposes here, we leave fractions in their improper form.)

This can be seen in the proportion with ratios, milk **:** pancakes. $\dfrac{\frac{2}{3}}{4} = \dfrac{\frac{4}{3}}{8}$

Also, it is possible that a solution is a fraction. If the solution is an improper fraction, we can leave it as a simplified improper fraction.

Example 4: Solve for the variable in each proportion.

a) $\dfrac{y}{5} = \dfrac{\frac{8}{5}}{4}$ b) $\dfrac{4}{4x-6} = \dfrac{3}{x}$

Procedure: Use cross-multiplication to get the equation into a linear form.

Answer:

a) $\dfrac{y}{5} = \dfrac{\frac{8}{5}}{4}$ Cross-multiply.

$y \cdot 4 = 5 \cdot \dfrac{8}{5}$ $\dfrac{5}{1} \cdot \dfrac{8}{5} = 8$

$4y = 8$ Divide each side by 4.

$\dfrac{4y}{\mathbf{4}} = \dfrac{8}{\mathbf{4}}$

$y = 2$ ↗

Verify $y = 2$:

$\dfrac{2}{5} \overset{?}{=} \dfrac{\frac{8}{5}}{4}$

(Treat the right side as $\frac{8}{5} \div 4$.)

$\dfrac{2}{5} = \dfrac{8}{5} \div 4$

$\dfrac{2}{5} = \dfrac{8}{5} \cdot \dfrac{1}{4}$

$\dfrac{2}{5} = \dfrac{8}{20}$

$\dfrac{2}{5} = \dfrac{2}{5}$

$\dfrac{2}{5} = \dfrac{2}{5}$ ✓

b) $\dfrac{4}{4x-6} = \dfrac{3}{x}$ Show parentheses in the left side denominator.

$\dfrac{4}{(4x-6)} = \dfrac{3}{x}$ Cross-multiply.

$4 \cdot x = (4x-6) \cdot 3$ Distribute.

$4x = 12x - 18$ Add $-12x$ to each side.

$\underline{\mathbf{-12x = -12x}}$

$-8x = -18$ Divide each side by -8.

$\dfrac{-8x}{\mathbf{-8}} = \dfrac{-18}{\mathbf{-8}}$ Simplify.

$x = \dfrac{9}{4}$ Verify $x = \frac{9}{4}$. ↓

Here is the work involved in verifying the solution for Example 4b): **Verify $x = \frac{9}{4}$:**

$$\frac{4}{4\cdot\left(\frac{9}{4}\right) - 6} \stackrel{?}{=} \frac{3}{\frac{9}{4}}$$

The left side denominator simplifies to $4\cdot\left(\frac{9}{4}\right) - 6 = 9 - 6 = 3$.

To evaluate the right side, think of the fraction as $3 \div \frac{9}{4}$. This can be evaluated as $3 \div \frac{9}{4} = \frac{3}{1} \div \frac{9}{4} = \frac{3}{1} \cdot \frac{4}{9} = \frac{12}{9} = \frac{4}{3}$.

$$\frac{4}{3} \stackrel{?}{=} \frac{3}{\frac{9}{4}}$$

$$\frac{4}{3} = \frac{4}{3} \quad \checkmark$$

Note: In a proportion, when the solution is a fraction, it is often more efficient to verify that the work was done correctly than to verify the solution.

You Try It 4 Solve each proportion. Use Example 4 as a guide.

a) $\frac{\frac{8}{3}}{2} = \frac{p}{3}$

b) $\frac{5}{\frac{15}{4}} = \frac{8}{m}$

c) $\frac{3x + 2}{30} = \frac{x}{5}$

d) $\frac{3}{y} = \frac{36}{6y + 5}$

APPLICATIONS WITH PROPORTIONS: THE PROPORTION TABLE

A **rate** is a ratio in which the two parts, numerator and denominator, have a different unit of measure, such as a car's gas mileage, $\frac{20 \text{ miles}}{1 \text{ gallon}}$ (20 miles per gallon) or the cost of renting a bowling lane, $\frac{\$15}{2 \text{ hours}}$ ($15 for every two hours of bowling) .

If a rate is consistent, then we can use a proportion to answer questions about *scale*, or size. For example, on a road map you might find that every inch on the map is equivalent to 2 miles of actual road. This could be set up as a rate, 1 inch for every 2 miles:

$$\frac{1 \text{ inch}}{2 \text{ miles}}$$

This is one *set* of information.

We might be curious about a certain length of highway that, on the map, is 5 inches long. We would find that the actual road would be 10 miles long; this is a second set of information.

Let's look at the two sets of information as a proportion:

first set of information = second set of information

$$\frac{1 \text{ inch}}{2 \text{ miles}} = \frac{5 \text{ inches}}{10 \text{ miles}}$$

Numerically, we get: $\frac{1}{2} = \frac{5}{10}$

Cross-multiplying, we can see that these are equivalent ratios:

$$1 \cdot 10 = 2 \cdot 5$$
$$10 = 10$$

In working with rates (ratios) and proportions, consistency is the key. This means that:

1. the numerator of each fraction should be of the same units of measure (or of the same *count*);
2. the denominator of each fraction should be of the same units of measure (or of the same *count*); and
3. the smaller units stay together and the larger units stay together.

In comparing the inches on the map to the actual miles of the road, we can set up each ratio according to a **scheme**, such as $\frac{\text{inches}}{\text{miles}}$. (A *scheme* is a plan of approach; it gives guidelines on how to approach a problem.)

We can be sure that we are consistent by setting up a table to outline our scheme, as shown here:

This table uses the scheme we've established, $\frac{\text{inches}}{\text{miles}}$: we put inches in the numerator, and the corresponding miles in the denominator.

Scheme:	smaller units (first set)	larger units (second set)
inches	1	5
miles	2	10

The table indicates that 1 inch will be paired with 2 miles and 5 inches will be paired with 10 miles, so we can get the proportion directly from the table:

$$\frac{1 \text{ inch}}{2 \text{ miles}} = \frac{5 \text{ inches}}{10 \text{ miles}} \quad \text{or} \quad \text{just} \quad \frac{1}{2} = \frac{5}{10}$$

In general, to solve a proportion problem:

1. Set up a scheme based on the information given; write a legend identifying the unknown value.
2. Set up a proportion table based on that scheme.
3. Set up a proportion based on the proportion table.
4. Solve the proportion and write a concluding sentence.

Example 5: 4 trucks are needed to haul away 10 tons of trash in a day. How many trucks are needed to haul away 15 tons of trash in a day?

Procedure: The first set of information is 4 trucks and 10 tons; the second set is x trucks and 15 tons. Let's use a scheme of $\frac{\text{trucks}}{\text{tons}}$. Set up the legend and the proportion table.

Legend: Let x = the number of trucks needed for 15 tons of trash.

Scheme:	smaller units	larger units
trucks	4	x
tons	10	15

In the table, place a fraction bar between each numerator and denominator, and place an equal sign between the fractions. This is the proportion.

Answer:

$\frac{4}{10} = \frac{x}{15}$	Cross-multiply.	An optional first step is to simplify the fraction on the left.
$4 \cdot 15 = 10 \cdot x$	Multiply.	
$60 = 10x$	Divide each side by 10.	
$\frac{60}{\mathbf{10}} = \frac{10x}{\mathbf{10}}$	Simplify.	
$6 = x$		

Concluding sentence: 6 trucks are needed to haul away 15 tons of trash.

For the following You Try It exercises, use Example 5 as a guide. Note that the variable, the unknown value, could be in the numerator or the denominator.

You Try It 5 Miguel earns \$50 for 4 hours of work. How much does he earn for 6 hours of work?

Legend ____

Scheme:

Sentence: ____

You Try It 6 4 adults are needed to supervise 18 children on a field trip. How many adults are needed to supervise 45 children on a field trip?

Legend ____

Scheme:

Sentence: ____

You Try It 7 On a map of Germany, 6 inches represents 80 km. How many inches represent 200 kilometers?

Legend ____

Scheme:		

Sentence: ____

You Try It 8 Tunde's Jeep can travel 120 miles on 9 gallons of gas. How many gallons of gas are required for his Jeep to travel 280 miles?

Legend ____

Scheme:		

Sentence: ____

You Try It Answers

You Try It 1: a) $150 \neq 120$ not equivalent   b) $72 = 72$ equivalent   c) $5 = 5$ equivalent

You Try It 2: a) $x = 6$ b) $x = 12$ c) $x = 21$

You Try It 3: a) $x = 2$ b) $y = 4$ c) $w = 9$

You Try It 4: a) $p = 4$ b) $m = 6$ c) $x = \frac{2}{3}$ d) $y = \frac{5}{6}$

You Try It 5: Miguel earns $75 for 6 hours of work.

You Try It 6: 10 adults are needed to supervise 45 children.

You Try It 7: 15 inches represent 200 kilometers

You Try It 8: 21 gallons of gas are required for his Jeep to travel 280 miles

Section 2.5 Exercises

Think Again.

1. Can we use cross-multiplication to solve the equation $\frac{x+1}{4} + 7 = \frac{x}{3}$? Why or why not?

2. If two ratios are not equivalent, is it still possible to set up a proportion between them? Explain your answer or show an example that supports your answer.

Focus Exercises.

Use cross-multiplication to determine whether the pair of ratios is equivalent or not.

3. $\frac{18}{12} \stackrel{?}{=} \frac{3}{2}$ **4.** $\frac{4}{12} \stackrel{?}{=} \frac{5}{10}$ **5.** $\frac{12}{30} \stackrel{?}{=} \frac{8}{15}$ **6.** $\frac{20}{30} \stackrel{?}{=} \frac{8}{12}$

Solve each proportion. Write any improper fraction answer as a mixed number.

7. $\frac{2}{3} = \frac{x}{6}$

8. $\frac{w}{6} = \frac{21}{14}$

9. $\frac{18}{v} = \frac{15}{10}$

10. $\frac{21}{12} = \frac{x}{8}$

11. $\frac{8}{60} = \frac{m}{45}$

12. $\frac{\frac{1}{2}}{x} = \frac{1}{8}$

13. $\frac{w+1}{5} = \frac{3}{15}$

14. $\frac{11}{7} = \frac{4-3y}{14}$

15. $\frac{3}{14} = \frac{x}{21}$

16. $\frac{5x+2}{30} = \frac{x}{3}$

17. $\frac{2y-4}{2} = \frac{3y}{5}$

18. $\frac{2p+7}{4p} = \frac{3}{4}$

19. $\frac{y}{8} = \frac{3y+10}{4}$

20. $\frac{2}{m} = \frac{7}{4m+1}$

21. $\frac{5y-12}{6} = \frac{5y-1}{8}$

22. $\frac{9}{4} = \frac{6-3r}{2r+1}$

23. $\frac{\frac{11}{2}}{m} = \frac{11}{6}$

24. $\frac{p}{9} = \frac{\frac{25}{3}}{5}$

25. $\frac{10}{y} = \frac{3}{\frac{36}{5}}$

26. $\frac{\frac{32}{5}}{2} = \frac{8n}{5}$

For each of the following, set up a proportion table. Write a legend and solve the corresponding proportion. Write a sentence answering the question.

27. Melissa is traveling in London. She can exchange \$5 (American Dollars) for £3 (British Pounds). How many pounds will she get for \$65?

28. At a candy store, the price of 12 inches of licorice is 45¢. What is the price for 20 inches of licorice?

29. Tom received 18 votes for every 30 people who voted. If a total of 160 people voted, how many votes did Tom receive?

30. Cheyenne was able to drive 327 miles on 15 gallons of gasoline. At that rate, how many miles will he be able to drive on 25 gallons?

31. A flooring company sells flooring material for \$18 per square yard. How many square yards of flooring material can be purchased for \$270?

32. At a hardware store, the price of 8 feet of chain is \$3.40. What length of chain will \$8.50 buy?

33. Delon can ride his bike an average of 12 miles in 45 minutes. At that rate, how many minutes will it take him to ride 20 miles?

34. Sandy's car can travel 132 miles on 8 gallons of gas. How far can her car travel on 12 gallons of gas?

35. Bryan can run an average of 6 miles in 32 minutes. How far can he run in 48 minutes?

36. At Charles County Community College, there are 10 math classes for every 25 English classes. If the college offers 38 math classes, how many English classes does it offer?

37. At the gym, Chandra's favorite exercise is a workout on the cross-trainer. He has found that he burns 265 calories in a 20-minute workout. How many minutes will Chandra need to work out on the cross-trainer to burn 1,060 calories?

38. At the gym, Luana's favorite exercise is a workout on the elliptical machine. She has found that she burns 425 calories in a 50-minute workout. How many calories will Luana burn in 70 minutes on the elliptical machine?

Think Outside the Box:

Solve each proportion for x.

39. $\dfrac{x^2 - x - 5}{x} = \dfrac{2x + 3}{2}$

40. $\dfrac{2x^2 + x - 10}{2x} = \dfrac{3x + 4}{3}$

41. $\dfrac{x}{x^2 - x + 12} = \dfrac{2}{2x - 5}$

42. $\dfrac{x + 4 - x^2}{5 - 3x} = \dfrac{x}{3}$